

Fibonacci Numbers: $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$

$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55, \dots$

Lucas Numbers: $L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n$

$L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, L_5 = 11, L_6 = 18, L_7 = 29, L_8 = 47, L_9 = 76, L_{10} = 123, \dots$

(1) **Sum of Fibonacci Numbers:** $\sum_{i=1}^n F_i = F_{n+2} - 1$

$$F_1 = F_3 - F_2$$

$$F_2 = F_4 - F_3$$

$$F_3 = F_5 - F_4$$

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$$F_n = F_{n+2} - F_{n+1}$$

Adding up the columns we get $\sum_{i=1}^n F_i = F_{n+2} - F_2 = F_{n+2} - 1$.

(2) **Binet's Formula (1843):** $F_n = \frac{\phi^n - \tau^n}{\sqrt{5}}$, where $\phi = \frac{1 + \sqrt{5}}{2}$ and $\tau = \frac{1 - \sqrt{5}}{2}$.

The golden ratio $\phi = \frac{1 + \sqrt{5}}{2}$ and $\tau = 1 - \phi = \frac{1 - \sqrt{5}}{2}$ are the roots of the equation $x^2 - x - 1 = 0$

Let $P(n)$ be the statement $F_n = \frac{\phi^n - \tau^n}{\sqrt{5}}$.

$$\text{When } n = 1, \text{ RHS} = \frac{\phi - \tau}{\sqrt{5}} = \frac{\frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = F_1 = \text{LHS.}$$

$$\text{When } n = 2, \text{ RHS} = \frac{\phi^2 - \tau^2}{\sqrt{5}} = \frac{\frac{1 + 2\sqrt{5} + 5}{4} - \frac{1 - 2\sqrt{5} + 5}{4}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = F_2 = \text{LHS.}$$

So $P(1)$ and $P(2)$ are true.

Assume that $P(k)$ and $P(k + 1)$ are true for some $k \in \mathbb{Z}^+$, i.e. $F_k = \frac{\phi^k - \tau^k}{\sqrt{5}}$, $F_{k+1} = \frac{\phi^{k+1} - \tau^{k+1}}{\sqrt{5}}$.

Prove that $P(k + 2)$ is true too, i.e. $F_{k+2} = \frac{\phi^{k+2} - \tau^{k+2}}{\sqrt{5}}$.

$$\begin{aligned} \text{LHS} &= F_{k+2} \\ &= F_k + F_{k+1} \\ &= \frac{\phi^k - \tau^k}{\sqrt{5}} + \frac{\phi^{k+1} - \tau^{k+1}}{\sqrt{5}} \\ &= \frac{\phi^k(1 + \phi) - \tau^k(1 + \tau)}{\sqrt{5}} \\ &= \frac{\phi^{k+2} - \tau^{k+2}}{\sqrt{5}} = \text{RHS} \end{aligned}$$

Since $P(1)$ and $P(2)$ are true, and $P(k)$ and $P(k + 1)$ are true $\Rightarrow P(k + 2)$ is true, by Math Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

(3) Cassini's Formula (1680): $F_{n-1} F_{n+1} - F_n^2 = (-1)^n$

Let $P(n)$ be the statement $F_{n-1} F_{n+1} - F_n^2 = (-1)^n$.

When $n = 1$, LHS = $F_0 F_2 - F_1^2 = 0 \times 1 - 1^2 = -1 = (-1)^1 = \text{RHS}$.

So $P(1)$ is true.

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e. $F_{k-1} F_{k+1} - F_k^2 = (-1)^k$.

Prove that $P(k + 1)$ is true too, i.e. $F_k F_{k+2} - F_{k+1}^2 = (-1)^{k+1}$.

$$\begin{aligned} \text{LHS} &= F_k F_{k+2} - F_{k+1}^2 \\ &= F_k (F_{k+1} + F_k) - F_{k+1} (F_k + F_{k-1}) \\ &= F_k F_{k+1} + F_k^2 - F_{k+1} F_k - F_{k+1} F_{k-1} \\ &= F_k^2 - F_{k+1} F_{k-1} \\ &= -(F_{k+1} F_{k-1} - F_k^2) \\ &= -(-1)^k \\ &= (-1)^{k+1} \\ &= \text{RHS} \end{aligned}$$

Since $P(1)$ is true, and $P(k)$ is true $\Rightarrow P(k + 1)$ is true, by Math Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

(4) Sum of Squares of Fibonacci Numbers: $\sum_{i=1}^n F_i^2 = F_n F_{n+1}$

$$\begin{aligned} F_1^2 &= F_1 (F_2 - F_0) = F_1 F_2 - F_1 F_0 \\ F_2^2 &= F_2 (F_3 - F_1) = F_2 F_3 - F_2 F_1 \\ F_3^2 &= F_3 (F_4 - F_2) = F_3 F_4 - F_3 F_2 \\ &\vdots \\ &\vdots \\ F_n^2 &= F_n (F_{n+1} - F_{n-1}) = F_n F_{n+1} - F_n F_{n-1} \end{aligned}$$

Adding up the columns we get $\sum_{i=1}^n F_i^2 = F_n F_{n+1} - F_1 F_0 = F_n F_{n+1}$.

Geometrical proof:

1	1	3	8
2			
5			

(5) Sum of Fibonacci Numbers with Even and Odd indices: $F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n}$
 $F_2 + F_4 + F_6 + \dots + F_{2n} = F_{2n+1} - 1$

$$\begin{aligned} F_1 &= F_2 \\ F_3 &= F_4 - F_2 \\ F_5 &= F_6 - F_4 \end{aligned}$$

$$\begin{aligned} & \vdots \\ & \vdots \\ & F_{2n-1} = F_{2n} - F_{2n-2} \end{aligned}$$

Adding up the columns, we get $F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n}$.

$$\begin{aligned} F_2 &= F_3 - F_1 \\ F_4 &= F_5 - F_3 \\ F_6 &= F_7 - F_5 \\ & \vdots \\ & \vdots \\ F_{2n} &= F_{2n+1} - F_{2n-1} \end{aligned}$$

Adding up the columns, we get $F_2 + F_4 + F_6 + \dots + F_{2n} = F_{2n+1} - 1$.

(6) Sum of Products of Consecutive Fibonacci Numbers $F_1F_2 + F_2F_3 + \dots + F_{2n-1}F_{2n} = F_{2n}^2$

$$\begin{aligned} F_1F_2 &= F_2^2 \\ F_2F_3 + F_3F_4 &= F_3(F_2 + F_4) = (F_4 - F_2)(F_4 + F_2) = F_4^2 - F_2^2 \\ F_4F_5 + F_5F_6 &= F_5(F_4 + F_6) = (F_6 - F_4)(F_6 + F_4) = F_6^2 - F_4^2 \\ & \vdots \\ & \vdots \end{aligned}$$

$$F_{2n-2}F_{2n-1} + F_{2n-1}F_{2n} = F_{2n-1}(F_{2n-2} + F_{2n}) = (F_{2n} - F_{2n-2})(F_{2n} + F_{2n-2}) = F_{2n}^2 - F_{2n-2}^2$$

Adding up the columns, we get $F_1F_2 + F_2F_3 + \dots + F_{2n-1}F_{2n} = F_{2n}^2$.

(7) Fibonacci Matrix: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$

Let $P(n)$ be the statement $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

When $n = 1$, $LHS = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = RHS$.

So $P(1)$ is true.

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$.

Prove that $P(k+1)$ is true too, i.e. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$.

$$\begin{aligned} LHS &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{pmatrix} \\ &= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} \\ &= RHS \end{aligned}$$

Since $P(1)$ is true, and $P(k)$ is true $\Rightarrow P(k+1)$ is true, by Math Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

(8) $F_{n-1} + F_{n+1} = L_n$

Let $P(n)$ be the statement $F_{n-1} + F_{n+1} = L_n$.

When $n = 1$, $LHS = F_0 + F_2 = 1 = L_1 = RHS$.

When $n = 2$, $LHS = F_1 + F_3 = 1 + 2 = 3 = L_2 = RHS$.

So P(1) and P(2) are true.

Assume that P(k) and P(k+1) are true for some $k \in \mathbb{Z}^+$,

$$\text{i.e. } F_{k-1} + F_{k+1} = L_k \text{ and } F_k + F_{k+2} = L_{k+1} .$$

Prove that P(k+2) is true too, i.e. $F_{k+1} + F_{k+3} = L_{k+2}$.

$$\begin{aligned} \text{LHS} &= F_{k+1} + F_{k+3} \\ &= F_{k-1} + F_k + F_{k+1} + F_{k+2} \\ &= L_k + L_{k+1} \\ &= L_{k+2} \\ &= \text{RHS} \end{aligned}$$

Since P(1) and P(2) are true, and P(k) and P(k+1) are true \Rightarrow P(k+2) is true, by Math Induction, P(n) is true for all $n \in \mathbb{Z}^+$.

$$(9) \mathbf{F_{m+n} = F_{m-1} F_n + F_m F_{n+1} .}$$

We fix m and prove by Mathematical Induction on n.

Let P(n) be the statement $F_{m+n} = F_{m-1} F_n + F_m F_{n+1}$.

When $n = 1$, LHS = $F_{m+1} = F_{m-1} + F_m = F_{m-1} F_1 + F_m F_2 = \text{RHS}$.

So P(1) is true.

Assume that P(n) is true for $n = 1, 2, 3, \dots, k$, and we will show that P(k+1) is true.

By assumption, $F_{m+k} = F_{m-1} F_k + F_m F_{k+1}$

$$\text{and } F_{m+(k-1)} = F_{m-1} F_{k-1} + F_m F_k .$$

Adding the two we get $F_{m+k} + F_{m+(k-1)} = F_{m-1} (F_k + F_{k-1}) + F_m (F_{k+1} + F_k)$

$$\Rightarrow F_{m+(k+1)} = F_{m-1} F_{k+1} + F_m F_{k+2} .$$

Since P(1) is true, and P(1), P(2),... P(k) are true \Rightarrow P(k+1) is true, by Math Induction, P(n) is true for all $n \in \mathbb{Z}^+$.

$$(10) \mathbf{F_{2n} = F_n L_n .}$$

$$\begin{aligned} F_{2n} &= F_{n+n} \\ &= F_{n-1} F_n + F_n F_{n+1} \quad \text{by (9) above} \\ &= F_n (F_{n-1} + F_{n+1}) \\ &= F_n L_n . \end{aligned}$$

(11) **Binet's Formula for Lucas Numbers:** $L_n = \phi^n + \tau^n$, where $\phi = \frac{1 + \sqrt{5}}{2}$ and $\tau = \frac{1 - \sqrt{5}}{2}$.

$$\begin{aligned} L_n &= F_{n-1} + F_{n+1} \\ &= \frac{\phi^{n-1} - \tau^{n-1}}{\sqrt{5}} + \frac{\phi^{n+1} - \tau^{n+1}}{\sqrt{5}} \\ &= \frac{\phi^n(1/\phi + \phi) - \tau^n(1/\tau + \tau)}{\sqrt{5}} \\ &= \frac{\phi^n \sqrt{5} - \tau^n (-\sqrt{5})}{\sqrt{5}} \\ &= \phi^n + \tau^n \end{aligned}$$

(12) **For any positive integers n and m, $F_n \mid F_{mn}$.**

We fix m and prove by Mathematical Induction on n.

Let P(n) be the statement $F_n \mid F_{mn}$.

For $n = 1$, clearly $F_m \mid F_m$ and so P(1) is true.

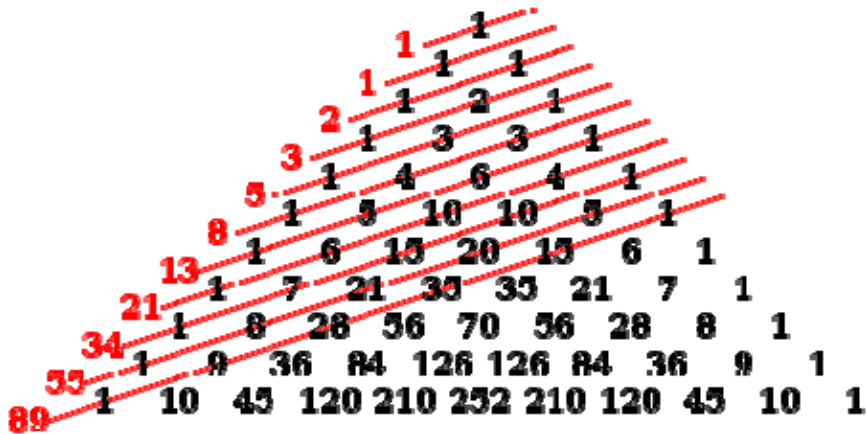
Assume that P(n) is true for $n = 1, 2, \dots, k$.

We will show that P(k+1) is true, i.e. $F_m \mid F_{m(k+1)}$

By (9) above, $F_{m(k+1)} = F_{m(k-1)} F_m + F_{mk} F_{m+1}$. By assumption $F_m \mid F_{mk}$, and so F_m divides the entire right side of the equation. Hence F_m divides $F_{m(k+1)}$.

Since $P(1)$ is true, and $P(1), P(2), \dots, P(k)$ are true $\Rightarrow P(k+1)$ is true, by Math Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

(13) The diagonals of Pascal's triangle sum to the Fibonacci numbers:
$$\sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i-1} = F_n$$



Let $P(n)$ be the statement
$$\sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i-1} = F_n.$$

For $n = 1$, $LHS = \binom{0}{0} = 1 = F_1 = RHS$.

$\therefore P(1)$ is true.

Assume that $P(n)$ is true for $n = 1, 2, \dots, k$ for some $k \in \mathbb{Z}^+$, i.e.

$$\binom{0}{0} = F_1$$

$$\binom{1}{0} = F_2$$

$$\binom{2}{0} + \binom{1}{1} = F_3$$

$$\binom{3}{0} + \binom{2}{1} = F_4$$

\vdots

$$\binom{k-2}{0} + \binom{k-3}{1} + \dots + \lfloor \frac{k}{2} \rfloor \text{ terms} = F_{k-1}$$

$$\binom{k-1}{0} + \binom{k-2}{1} + \dots + \lfloor \frac{k+1}{2} \rfloor \text{ terms} = F_k$$

To show that $P(k+1)$ is true, i.e.
$$\sum_{i=1}^{\lfloor \frac{k+2}{2} \rfloor} \binom{k+1-i}{i-1} = F_{k+1}.$$

$$\begin{aligned}
\text{LHS} &= \sum_{i=1}^{\lfloor \frac{k+2}{2} \rfloor} \binom{k+1-i}{i-1} \\
&= \binom{k}{0} + \binom{k-1}{1} + \binom{k-2}{2} + \dots \text{ to } \lfloor \frac{k+2}{2} \rfloor \text{ terms} \\
&= \binom{k-1}{0} + \binom{k-2}{0} + \binom{k-2}{1} + \binom{k-3}{1} + \binom{k-3}{2} + \dots \text{ to } \lfloor \frac{k+2}{2} \rfloor \text{ terms} \\
&= \binom{k-1}{0} + \binom{k-2}{1} + \binom{k-3}{2} + \dots + \binom{k-2}{0} + \binom{k-3}{1} + \dots \text{ to } \lfloor \frac{k+2}{2} \rfloor \text{ terms} \\
&= F_k + F_{k-1} \\
&= F_{k+1}
\end{aligned}$$

Since $P(1)$ is true, and $P(1), P(2), \dots, P(k)$ are true $\Rightarrow P(k+1)$ is true, by Math Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Other Identities:

$$\begin{aligned}
F_n + F_{n+3} &= 2F_{n+2} \\
F_n + 2F_{n+1} &= F_{n+3} \\
F_{n-2} + F_{n+2} &= 3F_n \\
F_{n-2} + F_n + F_{n+2} &= 4F_n \\
F_n + 2F_{n-1} &= L_n \\
F_n + L_n &= 2F_{n+1} \\
L_{n-1} + L_{n+1} &= 5F_n \\
2L_n + L_{n+1} &= 5F_{n+1} \\
2L_{n+1} - L_n &= 5F_n \\
F_{n-2} + F_{n+2} &= 3F_n \\
F_{n+2} - F_{n-2} &= L_n \\
L_{n+2} - L_{n-2} &= 5F_n \\
L_n + L_{n+3} &= 2L_{n+2} \\
F_n^2 + F_{n+1}^2 &= F_{2n+1} \\
F_{n+1}^2 - F_n^2 &= F_{n-1} F_{n+2} \\
F_{n+k}^2 - F_{n-k}^2 &= F_{2n} F_{2k} \\
L_{4n} - 2 &= 5F_{2n}^2 \\
L_{4n+2} + 2 &= 5F_{2n+1}^2 \\
L_{4n+2} - 2 &= L_{2n+1}^2 \\
L_{4n} + 2 &= L_{2n}^2 \\
L_{n+1}^2 - L_{n-1}^2 &= 5F_{2n} \\
L_n^2 + L_{n+1}^2 &= L_{2n} + L_{2n+2}
\end{aligned}$$

Catalan's Identity (1879)

$$\begin{aligned}
F_n^2 - F_{n+k} F_{n-k} &= (-1)^{n-k} F_k^2 \\
L_{n-1} L_{n+1} - L_n^2 &= 5(-1)^{n+1} \\
L_{n-k} L_{n+k} - L_n^2 &= 5(-1)^{n+k} F_k^2
\end{aligned}$$

D'Ocagne's Identity

$$F_m F_{n+1} + F_n F_{m+1} = (-1)^n F_{m-n}$$

Gelin-Cesaro Identity (1880)

$$F_{n-2} F_{n-1} F_{n+1} F_{n+2} + 1 = F_n^4$$

Candido's Identity (1951)

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 = 2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)$$

Aurifeuille's Identity (1879)

$$L_n (L_{2n} + 5F_n + 3) (L_{2n} - 5F_n + 3) = L_{5n}, \text{ for } n \text{ odd}$$