Functions of Complex Numbers

1. Powers and Exponential Function

If $z = re^{i\theta}$ where $r \in \mathbb{R}$ and $-\pi < \theta \leq \pi$, then the principal value of

$$\begin{aligned} z^{x+iy} &= (re^{i\theta})^{x+iy} \\ &= r^x r^{iy} e^{ix\theta} e^{-y\theta} \\ &= r^x e^{-y\theta} e^{iy \ln r} e^{ix\theta} \\ &= r^x e^{-y\theta} e^{i(x\theta + y \ln r)} \end{aligned}$$

$$\therefore z^{x+iy} = r^x e^{-y\theta} [\cos (x\theta + y \ln r) + i \sin (x\theta + y \ln r)].$$

In particular, $e^{x+iy} = e^x e^{iy}$

$$\therefore e^{x+iy} = e^x [\cos y + i \sin y].$$

Since z is also given by $z = re^{i(\theta+2k\pi)}$, where $k \in \mathbb{Z}$, the other values of z^{x+iy} are given by

$$z^{x+iy} = r^x e^{-y(\theta+2k\pi)} [\cos (x\theta + 2k\pi x + y \ln r) + i \sin (x\theta + 2k\pi x + y \ln r)].$$

Example 1

(i) Since $i = e^{i\pi/2}$ and $i = 0 + i1$, then $i^i = e^{-\pi/2} [\cos 0 + i \sin 0] = e^{-\pi/2} \approx 0.208$ (which, surprisingly, is a real number!)

(ii) Since $1 + i = \sqrt{2}\, e^{i\pi/4}$, then $(1 + i)^{1+i} = \sqrt{2}\, e^{-\pi/4} [\cos (\frac{\pi}{4} + \ln \sqrt{2}) + i \sin (\frac{\pi}{4} + \ln \sqrt{2})] \approx 0.274 + 0.584i$

(iii) $e^{1+i} = e^1 [\cos 1 + i \sin 1] \approx 1.47 + 2.29i$

(iv) $2^i = e^{i \ln 2} = e^0 [\cos (\ln 2) + i \sin (\ln 2)] \approx 0.769 + 0.639i$

2. Logarithm Function

Let $z = re^{i\theta}$, where $r = | z |$ and $\theta = \arg (z)$. Then the principal value of $\ln z$ is given by

$$\begin{aligned} \ln z &= \ln r + \ln e^{i\theta} \\ &= \ln r + i\theta. \end{aligned}$$

Since z is also given by $z = re^{i(\theta+2k\pi)}$, where $k \in \mathbb{Z}$, the other values of $\ln z$ are given by

$$\ln z = \ln r + i(\theta + 2k\pi) \quad \text{for all } k \in \mathbb{Z}.$$

Example 2

(i) Since $-1 = e^{i\pi}$, $\ln (-1) = i\pi$

(ii) Since $i = e^{i\pi/2}$, $\ln i = i\frac{\pi}{2}$

(iii) $\log_{10} (-1) = \frac{\ln (-1)}{\ln 10} = \frac{i\pi}{\ln 10}$

3. Trigonometric Functions

From $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$,

we obtain $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$.

We define the hyperbolic cosine and sine functions as $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$.

Immediately we see that $\cos (ix) = \frac{e^{-x} + e^x}{2} = \cosh x$ and $\sin (ix) = \frac{e^{-x} - e^x}{2i} = \frac{e^{-x} - e^x}{2i} \times \frac{i}{i} = i \sinh x$.

$$\begin{aligned} \therefore \cos (x + yi) &= \cos x \cos (yi) - \sin x \sin (yi) \\ &= \cos x \cosh y - i \sin x \sinh y \\ \& \sin (x + yi) &= \sin x \cos (yi) + \cos x \sin (yi) \\ &= \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

Example 3

(i) $\sin i = \sin (0 + i) = \sin 0 \cosh 1 + i \cos 0 \sinh 1$

$= i \sinh 1$

$= i \dfrac{e - e^{-1}}{2} \qquad = 1.18i$

(ii) $\cos i = \cos(0 + i)$

$= \cos 0 \cosh 1 - i \sin 0 \sinh 1$

$= \cosh 1$

$= \dfrac{e + e^{-1}}{2} \qquad = 1.54$

(iii) $\sin(1 + i) = \sin 1 \cosh 1 + i \cos 1 \sinh 1$

$= 1.30 + 0.635i$

4. Inverse Trigonometric Functions

Let $z = \cos w = \dfrac{e^{iw} + e^{-iw}}{2}$.

$2z = e^{iw} + e^{-iw}$

$2ze^{iw} = e^{2iw} + 1$

$e^{2iw} - 2ze^{iw} + 1 = 0$

$e^{iw} = \dfrac{2z \pm \sqrt{4z^2 - 4}}{2}$

$iw = \ln(z \pm \sqrt{z^2 - 1})$

$w = -i \ln(z \pm \sqrt{z^2 - 1})$

For the principal value, we use the positive sign. Hence

$$\cos^{-1} z = -i \ln(z + \sqrt{z^2 - 1})$$

Similarly let $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$.

$2iz = e^{iw} - e^{-iw}$

$2ize^{iw} = e^{2iw} - 1$

$e^{2iw} - 2ize^{iw} - 1 = 0$

$e^{iw} = \dfrac{2iz \pm \sqrt{-4z^2 + 4}}{2}$

$iw = \ln(iz \pm \sqrt{1 - z^2})$

$w = -i \ln(iz \pm \sqrt{1 - z^2})$

Using only the positive sign, we have

$$\sin^{-1} z = -i \ln(iz + \sqrt{1 - z^2})$$

Example 4

(i) $\cos^{-1} 2 = -i \ln(2 + \sqrt{3})$

(ii) $\sin^{-1} 2 = -i \ln(2i + \sqrt{-3})$

$= -i \ln[\, i(2 + \sqrt{3})\,]$

$= -i [\, \ln e^{i\pi/2} + \ln(2 + \sqrt{3})\,]$

$= -i [\, i\dfrac{\pi}{2} + \ln(2 + \sqrt{3})\,]$

$= \dfrac{\pi}{2} - i \ln(2 + \sqrt{3})$

(iii) $\cos^{-1} i = -i \ln(i + \sqrt{i^2 - 1})$

$= -i \ln(i + \sqrt{-2})$

$= -i \ln[\, i(1 + \sqrt{2})\,]$

$= -i [\, \ln e^{i\pi/2} + \ln(1 + \sqrt{2})\,]$

$= -i [\, i\dfrac{\pi}{2} + \ln(1 + \sqrt{2})\,]$

$= \dfrac{\pi}{2} - i \ln(1 + \sqrt{2})$