H2 MATHEMATICS SUMMARY BOOKLET

Trigonometric Formulas given in formula booklet:

$\sin (A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$

$\cos (A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$

$$\tan (A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$\sin 2x \equiv 2 \sin x \cos x$

$\cos 2x \equiv \cos^2 x - \sin^2 x$

$\equiv 2 \cos^2 x - 1 \equiv 1 - 2 \sin^2 x$

$$\tan 2x \equiv \frac{2 \tan x}{1 - \tan^2 x}$$

Principal values: $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$, $0 \le \cos^{-1} x \le \pi$, $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$

Trigonometric Formulas **NOT** given in formula booklet:

$\sin^2 x + \cos^2 x = 1$ $\quad 1 + \tan^2 x = \sec^2 x$ $\quad 1 + \cot^2 x = \operatorname{cosec}^2 x$

$\sec x = \frac{1}{\cos x}$ $\quad \operatorname{cosec} x = \frac{1}{\sin x}$ $\quad \cot x = \frac{1}{\tan x}$

$\sin\left(\frac{\pi}{2} - x\right) = \cos x$ $\quad \cos\left(\frac{\pi}{2} - x\right) = \sin x$ $\quad \tan\left(\frac{\pi}{2} - x\right) = \cot x$

Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine Rule: $a^2 = b^2 + c^2 - 2bc \cos A$

R-formulas: $a \sin x + b \cos x = R \sin (x + \alpha)$ where $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1} \frac{b}{a}$

$a \sin x - b \cos x = R \sin (x - \alpha)$

$a \cos x - b \sin x = R \cos (x + \alpha)$

Factor formulas:

$$\sin P + \sin Q \equiv 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

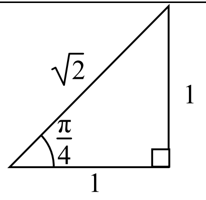
$$\sin P - \sin Q \equiv 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

$$\cos P + \cos Q \equiv 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$\cos P - \cos Q \equiv -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$ $\quad 2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

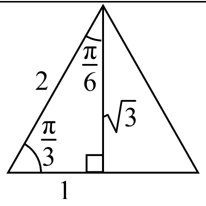
$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$ $\quad -2 \sin A \sin B = \cos (A + B) - \cos (A - B)$



$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

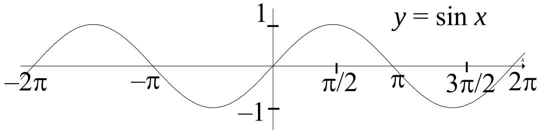
$\tan \frac{\pi}{4} = 1$

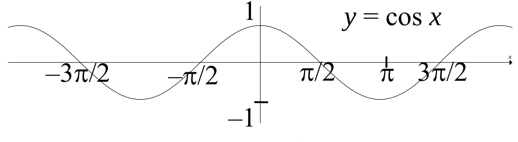


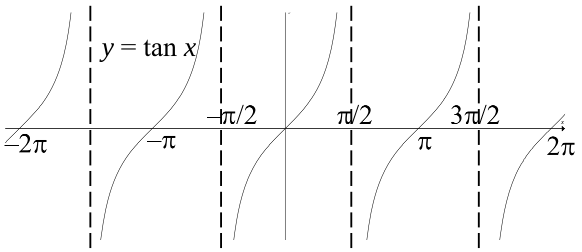
$\sin \frac{\pi}{6} = \frac{1}{2}$ $\quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

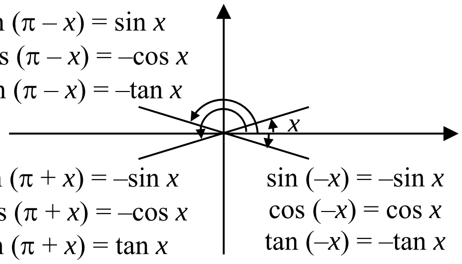
$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad \cos \frac{\pi}{3} = \frac{1}{2}$

$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad \tan \frac{\pi}{3} = \sqrt{3}$









$\sin (\pi - x) = \sin x$

$\cos (\pi - x) = -\cos x$

$\tan (\pi - x) = -\tan x$

$\sin (\pi + x) = -\sin x$

$\cos (\pi + x) = -\cos x$

$\tan (\pi + x) = \tan x$

$\sin (-x) = -\sin x$

$\cos (-x) = \cos x$

$\tan (-x) = -\tan x$

PARTIAL FRACTIONS SUMMARY

Make sure that the degree of the numerator is less than the degree of the denominator. Make sure that the denominator is completely factorised.

Distinct Linear Factors

Let $\dfrac{5-x}{(x^2-1)(x-2)} = \dfrac{5-x}{(x+1)(x-1)(x-2)}$ ← Factorise the denominator completely

$= \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{x-2}$

$5 - x = A(x-1)(x-2) + B(x+1)(x-2) + C(x^2-1)$

Substitute $x = -1$: $6 = 6A \Rightarrow A = 1$

Substitute $x = 1$: $4 = -2B \Rightarrow B = -2$

Substitute $x = 2$: $3 = 3C \Rightarrow C = 1$

$\therefore \dfrac{5-x}{(x^2-1)(x-2)} = \dfrac{1}{x+1} - \dfrac{2}{x-1} + \dfrac{1}{x-2}$

Let $\dfrac{2x^2-5}{(x-1)(x-2)} = 2 + \dfrac{A}{x-1} + \dfrac{B}{x-2}$ ← If the fraction is improper, divide first

$2x^2 - 5 = 2(x-1)(x-2) + A(x-2) + B(x-1)$

Substitute $x = 1$: $-3 = -A \Rightarrow A = 3$

Substitute $x = 2$: $3 = B$

$\therefore \dfrac{2x^2-5}{(x-1)(x-2)} = 2 + \dfrac{3}{x-1} + \dfrac{3}{x-2}$

Repeated Linear Factors

Let $\dfrac{5-3x}{(x-1)^2(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2}$ ← Note that the numerator of $(x-1)^2$ is B

$5 - 3x = A(x-1)(x-2) + B(x-2) + C(x-1)^2$

Substitute $x = 1$: $2 = -B \Rightarrow B = -2$

Substitute $x = 2$: $-1 = C$

Compare coefficients of x^2: $0 = A + C = A - 1 \Rightarrow A = 1$

$\therefore \dfrac{5-3x}{(x-1)^2(x-2)} = \dfrac{1}{x-1} - \dfrac{2}{(x-1)^2} - \dfrac{1}{x-2}$

Distinct Quadratic Factors

Let $\dfrac{2x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$ ← Note that the numerator of $x^2 + 1$ is $Bx + C$

$2x = A(x^2+1) + (Bx+C)(x-1)$

Substitute $x = 1$: $2 = 2A \Rightarrow A = 1$

Compare coefficients of x^2: $0 = A + B = 1 + B \Rightarrow B = -1$

Compare constants: $0 = A - C = 1 - C \Rightarrow C = 1$

$\therefore \dfrac{2x}{(x-1)(x^2+1)} = \dfrac{1}{x-1} + \dfrac{1-x}{x^2+1}$

SUMMARY OF SEQUENCES & SERIES

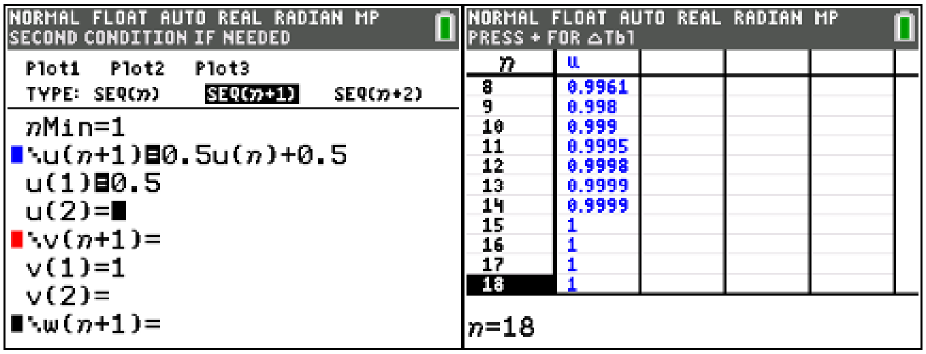
The nth term of a sequence $\boxed{u_n = S_n - S_{n-1}}$, where S_n = sum of first n terms of the sequence.

If $u_n \to L$ as $n \to \infty$ where $L \in \mathbb{R}$, the sequence is **convergent**. Otherwise it is **divergent**.

If $S_n \to L$ as $n \to \infty$ where $L \in \mathbb{R}$, the series is **convergent**. Otherwise it is **divergent**.

Example: Given $S_n = n + 0.5^n$.
As $n \to \infty$, $S_n \to \infty$. Hence S_n is divergent.
$u_n = n + 0.5^n - (n - 1 + 0.5^{n-1}) = 1 + 0.5^n - 0.5^{n-1} = 1 + 0.5^{n-1}(0.5 - 1) = 1 - 0.5^n$
As $n \to \infty$, $0.5^n \to 0$ and $u_n \to 1$. Hence $\{ u_n \}$ is convergent and $\lim\limits_{n\to\infty} u_n = 1$.

Example: $u_{n+1} = \dfrac{1}{2} u_n + \dfrac{1}{2}$ for $n \geq 1$, and $u_1 = \dfrac{1}{2}$.



```
NORMAL FLOAT AUTO REAL RADIAN MP
SECOND CONDITION IF NEEDED
  Plot1   Plot2   Plot3
  TYPE: SEQ(n)   SEQ(n+1)   SEQ(n+2)
nMin=1
 u(n+1)=0.5u(n)+0.5
 u(1)=0.5
 u(2)=
 v(n+1)=
 v(1)=1
 v(2)=
 w(n+1)=
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR ΔTbl
  n    u
  8    0.9961
  9    0.998
  10   0.999
  11   0.9995
  12   0.9998
  13   0.9999
  14   0.9999
  15   1
  16   1
  17   1
  18   1
n=18
```

By GC, $\{ u_n \}$ converges to 1.

ARITHMETIC PROGRESSIONS	GEOMETRIC PROGRESSIONS
nth term $u_n = a + (n - 1)d$.	nth term $u_n = ar^{n-1}$.
Sum to nth term $S_n = \dfrac{n}{2}[\, a + u_n \,]$	Sum to nth term $S_n = a\dfrac{1 - r^n}{1 - r} = a\dfrac{r^n - 1}{r - 1}$.
$= \dfrac{n}{2}[\, 2a + (n - 1)d \,]$.	Sum to infinity $S_\infty = \dfrac{a}{1 - r}$. S_∞ exists if $\lvert r \rvert < 1$.

To prove that a sequence u_1, u_2, u_3, ... is an A.P., show that $u_{n+1} - u_n$ = constant for all positive integer n. **Warning:** Do not just show that $u_3 - u_2 = u_2 - u_1$.

To show that a sequence $u_1, u_2, \ldots$ is a G.P., show that $\dfrac{u_{n+1}}{u_n}$ = constant for all positive integer n.

Example: The sum of n terms of a series is given by $S_n = n^2 + n$. Show that the terms form an A.P.

$$\begin{aligned} \text{The } n\text{th term } u_n = S_n - S_{n-1} &= n^2 + n - [(n - 1)^2 + (n - 1)] \\ &= n^2 + n - [n^2 - 2n + 1 + n - 1] \\ &= 2n \end{aligned}$$

$u_n - u_{n-1} = 2n - 2(n - 1) = 2$ = constant.
Hence the terms in the series form an A.P.

Example: A geometric sequence has first term $a > 0$ and common ratio 0.3 . Find the least integer n such that $\lvert S_n - S_\infty \rvert < 0.01a$.

$$\left| a\frac{1 - 0.3^n}{1 - 0.3} - \frac{a}{1 - 0.3} \right| < 0.01a$$

$$\left| \frac{1 - 0.3^n}{0.7} - \frac{1}{0.7} \right| < 0.01$$

$$0.3^n < 0.007$$

$$n > \frac{\log 0.007}{\log 0.3} = 4.12$$

$\therefore$ least $n = 5$.

Example: At the beginning of every month, John puts \$100 into a savings account which pays 2% interest monthly. How long would it take the account to reach \$6000?

Amount at the end of 1 month $= \$100 \times 1.02$

Amount at the end of 2 months $= \$(100 \times 1.02 + 100) \times 1.02$

$= 100(1.02^2) + 100(1.02)$

Amount at the end of 3 months $= \$[100(1.02^2) + 100(1.02) + 100] \times 1.02$

$= 100(1.02^3) + 100(1.02^2) + 100(1.02)$

$= 102\,\frac{1.02^3 - 1}{1.02 - 1} = 5100(1.02^3 - 1)$

Amount at the end of n months $= 5100(1.02^n - 1) \geq 6000$

$$1.02^n \geq \frac{37}{17}$$

$$n \geq 39.27$$

It would take 40 months to reach \$6000.

SUMMATION USING SIGMA NOTATION

$\sum_{r=1}^{n} a$, where a = constant $= a + a + ... + a$ (n times) $= na$	$\sum_{r=1}^{n} r$ $= 1 + 2 + ... + n$ $= \frac{n}{2}(n + 1)$	$\sum_{r=m}^{n} f(r) = f(m) + f(m + 1) + ... + f(n)$ $= \sum_{r=1}^{n} f(r) - \sum_{r=1}^{m-1} f(r)$

Example: $\sum_{r=0}^{n} (2r + 1)^2 = \sum_{r=0}^{n} (4r^2 + 4r + 1)$

$$= 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r + \sum_{r=0}^{n} 1$$

$$= 4\frac{n}{6}(n + 1)(2n + 1) + 4\frac{n}{2}(n + 1) + (n + 1)$$

$$= \frac{n + 1}{3}[\,2n(2n + 1) + 6n + 3\,]$$

$$= \frac{n + 1}{3}(4n^2 + 8n + 3)$$

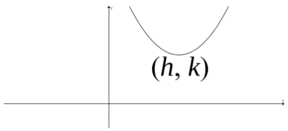
Replacing r by $r - 1$, $\sum_{r=-1}^{n} (2r + 3)^2 = \sum_{r-1=-1}^{r-1=n} [2(r - 1) + 3]^2$

$$= \sum_{r=0}^{n+1} (2r + 1)^2 = \frac{n + 2}{3}[4(n + 1)^2 + 8(n + 1) + 3]$$

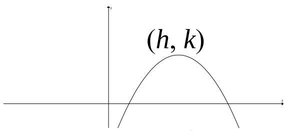
$$= \frac{n + 2}{3}(4n^2 + 16n + 15)$$

SUMMARY OF GRAPHS

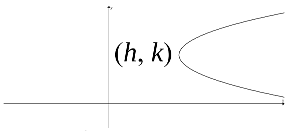
Parabolas



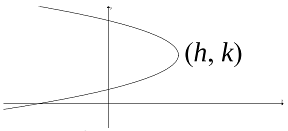
$y - k = p(x - h)^2,\ p > 0$



$y - k = p(x - h)^2,\ p < 0$



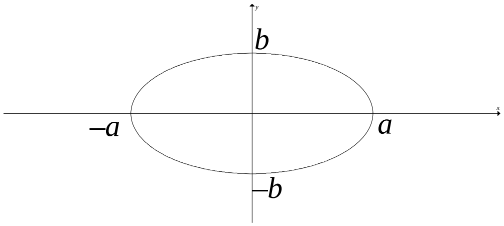
$(y - k)^2 = p(x - h),\ p > 0$



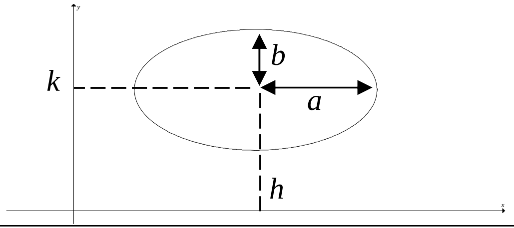
$(y - k)^2 = p(x - h),\ p < 0$

Ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

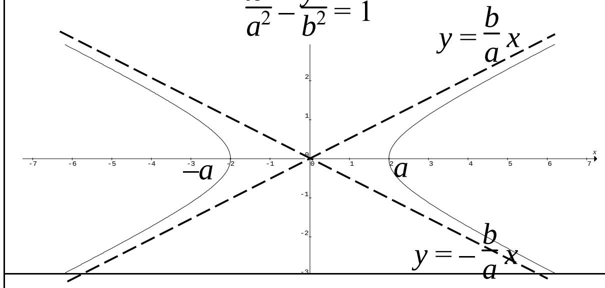


$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

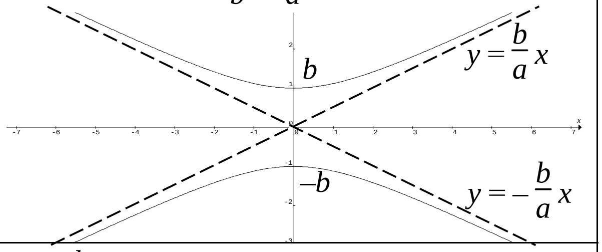


Hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

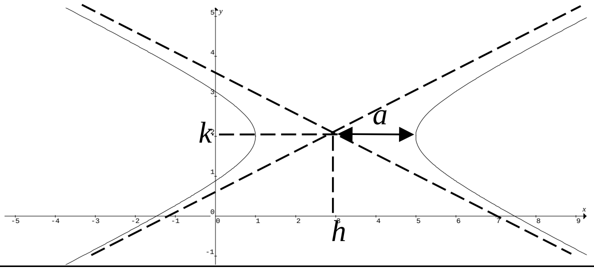


$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

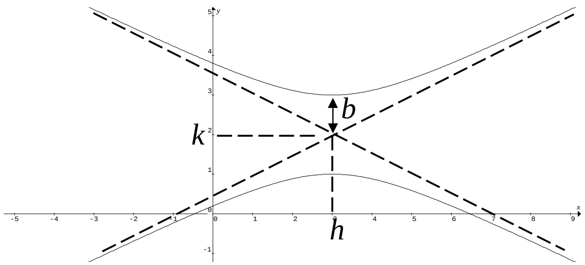


The asymptotes are $y = \pm\frac{b}{a}x$.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



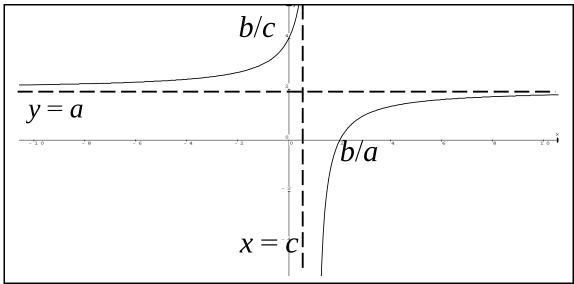
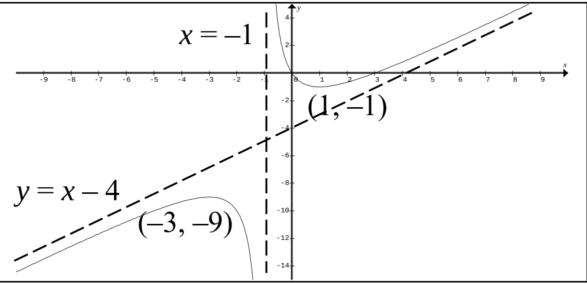
$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$



The asymptotes are obtained by solving $\frac{y-k}{b} = \pm\frac{x-h}{a}$, which gives $y = k \pm \frac{b}{a}(x - h)$.

Graphs of Rational Functions

If degree of numerator ≤ degree of denominator, the graph has a horizontal asymptote. E.g. $y = \frac{ax - b}{x - c}$ The vertical asymptote is $x = c$. The horizontal asymptote is $y = a$.	If degree of numerator is 1 more than degree of denominator, the graph has an oblique asymptote. E.g. $y = \frac{x^2 - 3x}{x + 1}$ $= x - 4 + \frac{4}{x + 1}$ by long division. The oblique asymptote is $y = x - 4$.

Let $y = \frac{x^2 - 3x}{x + 1}$

$xy + y = x^2 - 3x$

$x^2 - (y + 3)x - y = 0$

This quadratic equation has no real roots if $(y + 3)^2 + 4y < 0$

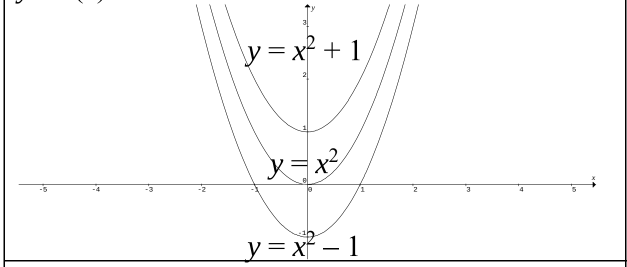
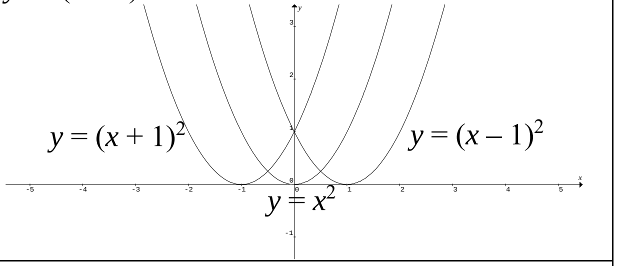
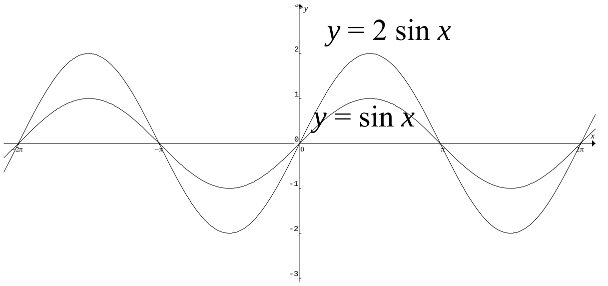
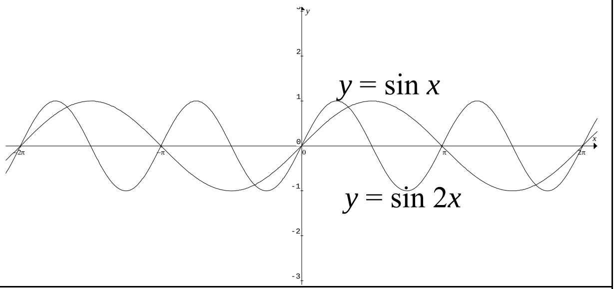
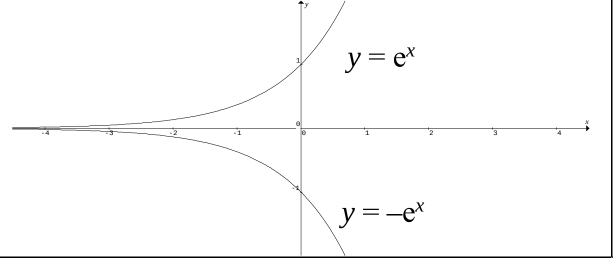
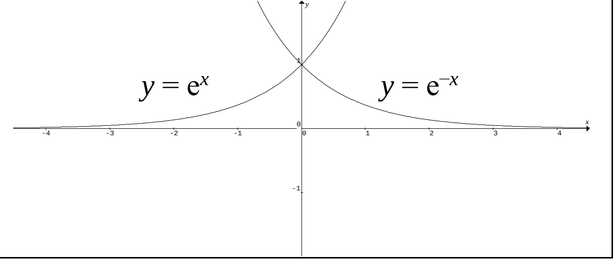
$$y^2 + 10y + 9 < 0$$

$$(y + 9)(y + 1) < 0$$

$$-9 < y < -1$$

Hence there are no points on the graph of $y = \frac{x^2 - 3x}{x + 1}$ for $-9 < y < -1$.

Transformation of Graphs

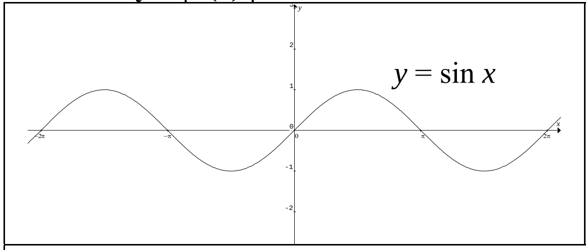
Translate by a units in direction of y–axis: $y = \mathrm{f}(x) + a$  	Translate by a units in direction of x–axis: $y = \mathrm{f}(x - a)$ 
Scale by factor a parallel to y–axis: $y = a\mathrm{f}(x)$  	Scale by factor $\frac{1}{a}$ parallel to x–axis: $y = \mathrm{f}(ax)$ 
Reflect along x–axis: $y = -\mathrm{f}(x)$  	Reflect along y–axis: $y = \mathrm{f}(-x)$ 

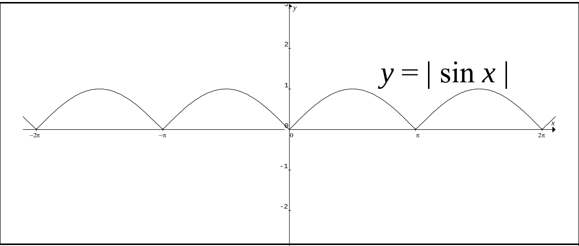
Order of Transformation

$$y = \text{f}(x) \xrightarrow[\text{in –ve } x \text{ direction}]{\text{translate by 1 unit}} \text{f}(x + 1) \xrightarrow[\text{by factor 2}]{\text{scale // } x\text{–axis}} \text{f}\left(\frac{x}{2} + 1\right)$$

$$y = \text{f}(x) \xrightarrow[\text{by factor 2}]{\text{scale // } x\text{–axis}} \text{f}\left(\frac{x}{2}\right) \xrightarrow[\text{in –ve } x \text{ direction}]{\text{translate by 1 unit}} \text{f}\left(\frac{x + 1}{2}\right)$$

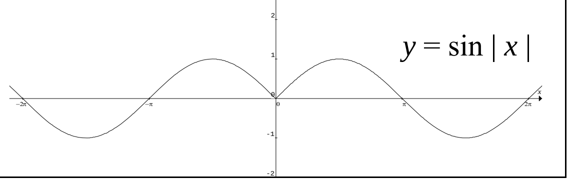
To draw y = | f(x) |:





To draw *y* = f (| *x* |):

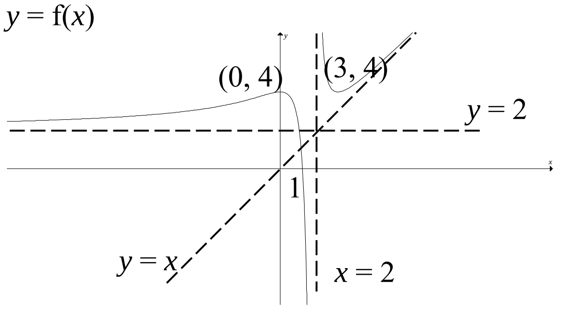
- Remove the part of the curve *y* = f(*x*) on the left of the *y*–axis.
- Copy the right hand side of the curve and reflect along the *y*–axis.

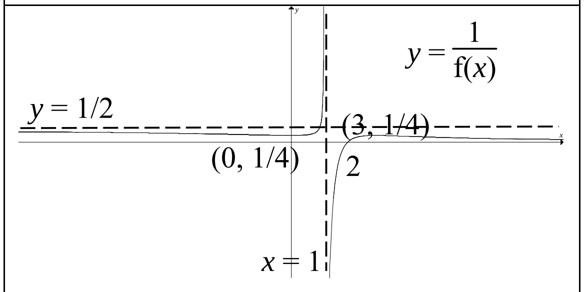


To draw $y = \dfrac{1}{\text{f}(x)}$:

- When f(*x*) = 0, $\dfrac{1}{\text{f}(x)}$ is undefined ⇒ *x*–intercepts become vertical asymptotes.
- As f(*x*) → ±∞, $\dfrac{1}{\text{f}(x)} \to 0 \Rightarrow$ vertical asymptotes become *x*–intercepts.
- Horizontal asymptote *y* = *a* becomes horizontal asymptote *y* = 1/*a*.
- Maximum point (*a*, f(*a*)) becomes minimum point (*a*, 1/f(*a*)), and vice versa.
- f(*x*) and $\dfrac{1}{\text{f}(x)}$ have the same sign.
- As f(*x*) increases, $\dfrac{1}{\text{f}(x)}$ decreases.

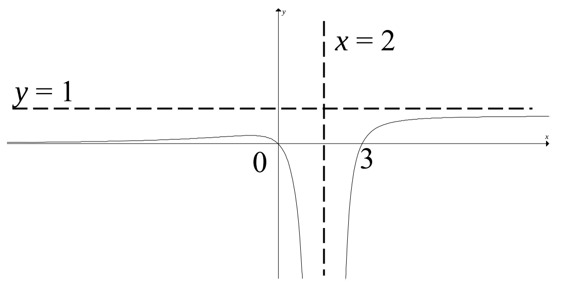
  As f(*x*) decreases, $\dfrac{1}{\text{f}(x)}$ increases.





To draw *y* = f ′(*x*)

- Vertical asymptotes remain the same.
- Horizontal asymptote *y* = *a* becomes horizontal asymptote *y* = 0.
- Oblique asymptote *y* = *ax* + *b* becomes horizontal asymptote *y* = *a*.
- Stationary point (*a*, f(*a*)) becomes *x*–intercept (*a*, 0).
- If *y* = f(*x*) is increasing, then f ′(*x*) > 0.
- If *y* = f(*x*) is decreasing, then f ′(*x*) < 0.



SUMMARY OF INEQUALITIES

Do not cancel common factors.

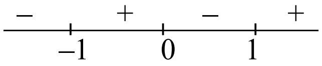
Do not cross-multiply a denominator unless the denominator is positive.

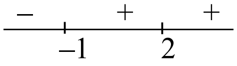
Do not square both sides of an inequality unless both sides are non-negative.

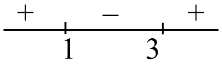
When multiplying both sides by a negative number, switch ‘<’ to ‘>’, or ‘≤’ to ‘≥’ (and vice versa).

Rearrange the inequality into the form $\frac{f(x)}{g(x)} < 0$ (or involving >, ≤ or ≥), and factorise f(x), g(x).

If unable to factorise f(x) or g(x), complete the square to check whether it is always positive.

<table>
<tr><th>Example of linear factors:</th><th>Example of repeated factors:</th><th>Example of a positive factor:</th></tr>
<tr>
<td>
$\frac{2}{x-1} \ge \frac{1}{x}$
$\frac{2}{x-1} - \frac{1}{x} \ge 0$
$\frac{2x-(x-1)}{(x-1)x} \ge 0$
$\frac{x+1}{(x-1)x} \ge 0$


$\therefore -1 \le x < 0$ or $x > 1$
</td>
<td>
$x^2 - 1 \ge (x + 1)(x^2 - 3x + 3)$
$(x+1)(x-1) \ge (x+1)(x^2 - 3x + 3)$
$(x + 1)(x^2 - 4x + 4) \le 0$
$(x + 1)(x - 2)^2 \le 0$


$\therefore x \le -1$ or $x = 2$
</td>
<td>
$\frac{x^2 - 4x + 5}{x^2 - 4x + 3} \ge 0$
$\frac{(x-2)^2 - 4 + 5}{(x-1)(x-3)} \ge 0$
$\frac{(x-2)^2 + 1}{(x-1)(x-3)} \ge 0$


$\therefore x < 1$ or $x > 3$
</td>
</tr>
</table>

The modulus function:

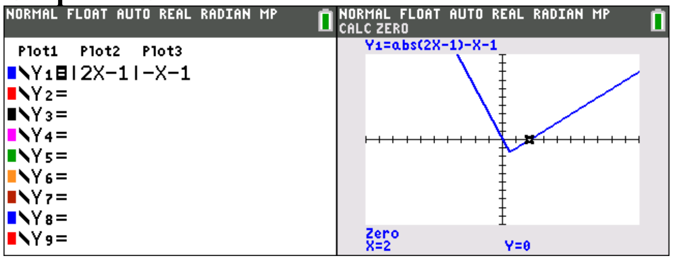
$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$|x| < a \Leftrightarrow -a < x < a$$

$$|x| > a \Leftrightarrow x < -a \text{ or } x > a$$

Example: Solve $|2x - 1| < x + 1$

Graphical Solution:



```
NORMAL FLOAT AUTO REAL RADIAN MP
 Plot1  Plot2  Plot3
■\Y1■|2X−1|−X−1
■\Y2=
■\Y3=
■\Y4=
■\Y5=
■\Y6=
■\Y7=
■\Y8=
■\Y9=
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
CALC ZERO
Y1=abs(2X-1)-X-1
Zero
X=2              Y=0
```

From the graph, $0 < x < 2$

Algebraic Solution:

$-(x + 1) < 2x - 1 < x + 1$

$-x - 1 < 2x - 1$ and $2x - 1 < x + 1$

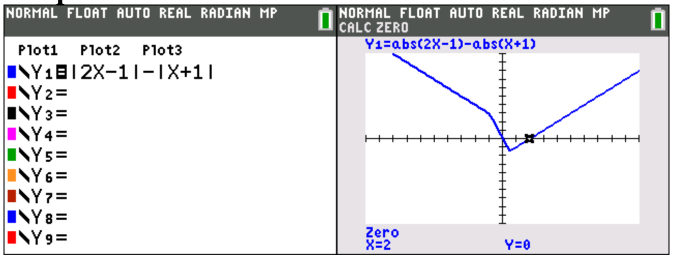
$3x > 0$ $\qquad\qquad$ $x < 2$

$x > 0$

Hence $0 < x < 2$

Example: Solve $|2x - 1| \ge |x + 1|$

Graphical Solution:



```
NORMAL FLOAT AUTO REAL RADIAN MP
 Plot1  Plot2  Plot3
■\Y1■|2X−1|−|X+1|
■\Y2=
■\Y3=
■\Y4=
■\Y5=
■\Y6=
■\Y7=
■\Y8=
■\Y9=
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
CALC ZERO
Y1=abs(2X-1)-abs(X+1)
Zero
X=2              Y=0
```

From the graph, $x \le 0$ or $x \ge 2$

Algebraic Solution:

$(2x - 1)^2 \ge (x + 1)^2$

$4x^2 - 4x + 1 \ge x^2 + 2x + 1$

$3x^2 - 6x \ge 0$

$x(x - 2) \ge 0$

$x \le 0$ or $x \ge 2$

Example of substitution: Solve $|2x^2 - 1| \ge |x^2 + 1|$

From above, $x^2 \le 0$ or $x^2 \ge 2$

$\therefore x = 0$ or $x \le -\sqrt{2}$ or $x \ge \sqrt{2}$.

EQUATIONS AND SYSTEMS OF LINEAR EQUATIONS

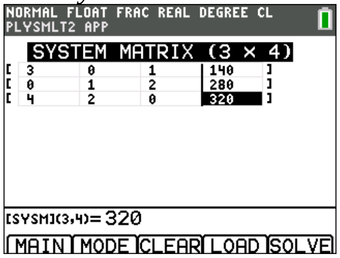
John bought 3 xylophones and 1 zither for \$140. Mary bought 1 yangqin and 2 zithers for \$280. Peter bought 4 xylophones and 2 yangqins for \$320. How much does each instrument cost?

Let x, y, z be the cost of each xylophone, yangqin and zither respectively.

$3x + z = 140$ ——(1)

$y + 2z = 280$ ——(2)

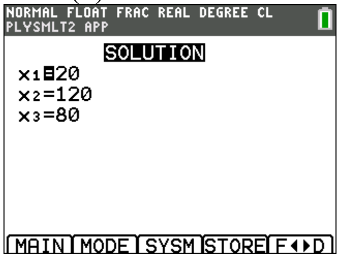
$4x + 2y = 320$ ——(3)



SYSTEM MATRIX (3 × 4)

3	0	1	140
0	1	2	280
4	2	0	320

[SYSM](3,4)= 320



SOLUTION

x_1=20
x_2=120
x_3=80

Each xylophone costs \$20, each yangqin costs \$120 and each zither costs \$80.

A cubic curve has stationary points at (1, 1) and (2, 0). Find the equation of the curve.

Let $y = ax^3 + bx^2 + cx + d$.

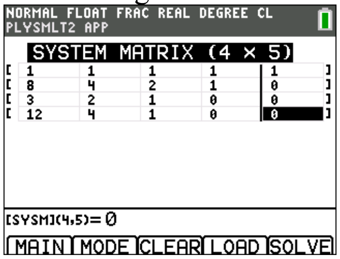
Substituting (1, 1): $a + b + c + d = 1$ ——(1)

Substituting (2, 0): $8a + 4b + 2c + d = 0$ ——(2)

$\frac{dy}{dx} = 3ax^2 + 2bx + c$.

Substituting $x = 1$: $3a + 2b + c = 0$ ——(3)

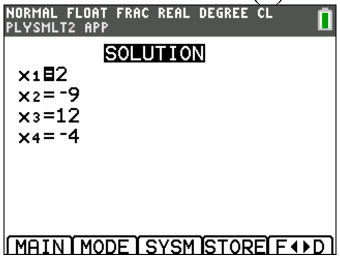
Substituting $x = 2$: $12a + 4b + c = 0$ ——(4)



SYSTEM MATRIX (4 × 5)

1	1	1	1	1
8	4	2	1	0
3	2	1	0	0
12	4	1	0	0

[SYSM](4,5)= 0



SOLUTION

x_1=2
x_2=⁻9
x_3=12
x_4=⁻4

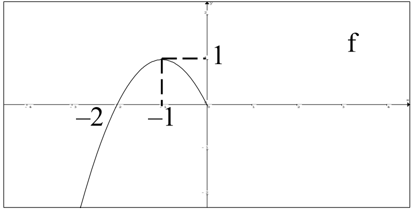
The equation of the curve is $y = 2x^3 - 9x^2 + 12x - 4$.

SUMMARY OF FUNCTIONS

INVERSE FUNCTIONS

For f^{-1} to exist, f must be one–to–one.	$D_{f^{-1}} = R_f$	$R_{f^{-1}} = D_f$

Let $f : x \to -x^2 - 2x, x \le 0$.



From the graph, we see that $R_f = (-\infty, 1]$.

Since the horizontal line $y = 0.5$ cuts the curve twice, f is not 1–1. Hence f^{-1} does not exist.

The largest domain such that f^{-1} exists is $(-\infty, -1]$.

To find f^{-1}, we let $y = f(x)$ and solve for x.

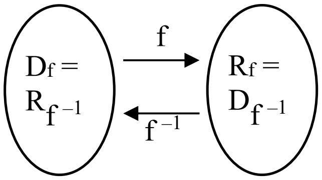
Let $y = -x^2 - 2x, x \le -1$

$x^2 + 2x + y = 0$

$$x = \frac{-2 \pm \sqrt{4-4y}}{2}$$

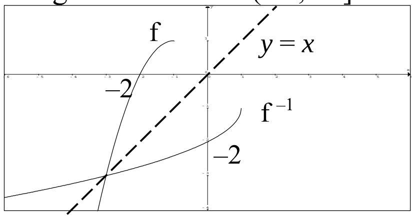
$$= -1 \pm \sqrt{1-y}$$

$$= -1 - \sqrt{1-y}\text{ , since } x \le -1$$



So $f^{-1} : x \to -1 - \sqrt{1-x}, x \le 1$.

Range of $f^{-1} = D_f = (-\infty, -1]$.



The graphs of f and f^{-1} are symmetrical about the line $y = x$.

To solve $f(x) = f^{-1}(x)$, let $-x^2 - 2x = x$

$x^2 + 3x = 0$

$x(x + 3) = 0$

$x = -3$ or $x = 0$ (reject since $x \le -1$)

COMPOSITE FUNCTIONS

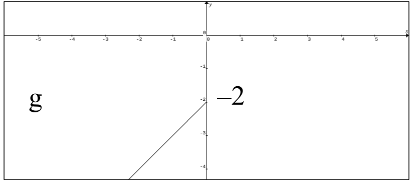
fg exists if $R_g \subseteq D_f$	$D_{fg} = D_g$	$D_g \xrightarrow{g} R_g \xrightarrow{f} R_{fg}$

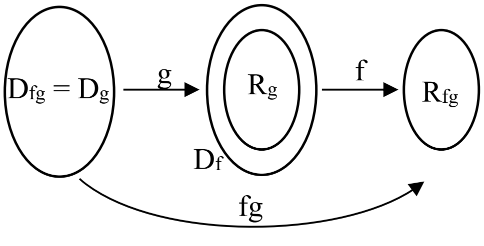
Let $f : x \to -x^2 - 2x, x \le -1$.

Let $g : x \to x - 2, x \le 0$.

Since $R_f = (-\infty, 1] \not\subseteq (-\infty, 0] = D_g$, gf does not exist.

gf exists if $R_f \subseteq D_g$





Since $R_g = (-\infty, -2] \subseteq (-\infty, -1] = D_f$, fg exists.

$fg : x \to -(x-2)^2 - 2(x-2) = 2x - x^2, x \le 0$.

$D_g = (-\infty, 0] \xrightarrow{g} (-\infty, -2] \xrightarrow{f} (-\infty, 0]$.

Hence $R_{fg} = (-\infty, 0]$.

SUMMARY OF DIFFERENTIATION

Product Rule: $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$ **Quotient Rule:** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Chain Rule: If $y = \mathrm{f}(u)$ and $u = \mathrm{g}(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$, e.g. $\frac{\mathrm{d}}{\mathrm{d}x}\sin^3 x^2 = 3\sin^2 x^2 (\cos x^2)\, 2x$

Trigonometric Functions: $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$ $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$ $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\mathrm{cosec}^2 x$ $\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$ $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{cosec}\, x = -\mathrm{cosec}\, x \cot x$

Exponential Functions: $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$ $\frac{\mathrm{d}}{\mathrm{d}x}a^x = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x\ln a} = \mathrm{e}^{x\ln a}\ln a = a^x \ln a$

Logarithmic Functions: $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$ $\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x}{\ln a} = \frac{1}{x\ln a}$

Inverse Trigonometric Functions (given in Formula List):

$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$ $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}$

Implicit Differentiation e.g. Differentiating $x^2 + y^3 = xy$ gives $2x + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$.

Parametric Differentiation: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$ e.g. if $y = \sin t$, $x = t^2$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\cos t}{2t}$.

Equation of tangent at the point (x_0, y_0) is $y - y_0 = m(x - x_0)$, where m = gradient at $x = x_0$.

Equation of normal at the point (x_0, y_0) is $y - y_0 = -\frac{1}{m}(x - x_0)$.

E.g. If $y = x^2$, then the equation of the tangent at (3, 9) is $y - 9 = 6(x - 3)$ i.e. $y = 6x - 9$

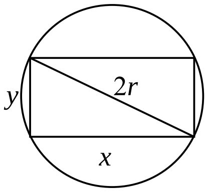
and the equation of the normal is $y - 9 = -\frac{1}{6}(x - 3)$ i.e. $y = -\frac{1}{6}x + \frac{19}{2}$.

Rate of Change: E.g. Let $V = \frac{4}{3}\pi r^3$, $\frac{\mathrm{d}r}{\mathrm{d}t} = 10$. When $r = 1$, $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2 10 = 40\pi$.

Maxima/Minima: Find the maximum area of a rectangle inscribed in a circle of radius r.

Area $A = xy = x\sqrt{(2r)^2 - x^2} \Rightarrow A^2 = x^2(4r^2 - x^2) = 4r^2x^2 - x^4$

$2A\frac{\mathrm{d}A}{\mathrm{d}x} = 8r^2x - 4x^3 = 4x(2r^2 - x^2) = 0 \Rightarrow x = r\sqrt{2}$

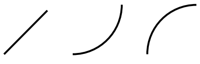


First Derivative Test:

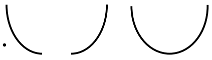
x	$r\sqrt{2}^-$	$r\sqrt{2}$	$r\sqrt{2}^+$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	+ve	0	–ve
	/	–	\

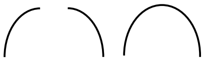
Second Derivative Test: $2A\frac{\mathrm{d}^2A}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}A}{\mathrm{d}x}\right)^2 = 8r^2 - 12x^2 = -16r^2 < 0$ when $x = r\sqrt{2} \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}x^2} < 0$

$\therefore$ maximum area $A = r\sqrt{2}\sqrt{4r^2 - 2r^2} = 2r^2$

Gradient: If $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$, then the curve is increasing. 

If $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$, then the curve is decreasing. 

Concavity: If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, then $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing and the curve is concave upwards. 

If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, then $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing and the curve is concave downwards. 

SUMMARY OF INTEGRATION TECHNIQUES

Integration by Standard forms:

$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c,$ for $n \neq -1$	$\int (Ax+B)^n \, \mathrm{d}x = \frac{1}{A} \frac{(Ax+B)^{n+1}}{n+1} + c,$ for $n \neq -1$	$\int \mathrm{f}'(x) \, [\mathrm{f}(x)]^n \, \mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c,$ for $n \neq -1$
$\int \frac{1}{x} \, \mathrm{d}x = \ln \lvert x \rvert + c$	$\int \frac{1}{Ax+B} \, \mathrm{d}x = \frac{1}{A} \ln \lvert Ax+B \rvert + c$	$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \, \mathrm{d}x = \ln \lvert \mathrm{f}(x) \rvert + c$
$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$	$\int \mathrm{e}^{Ax+B} \, \mathrm{d}x = \frac{1}{A} \mathrm{e}^{Ax+B} + c$	$\int \mathrm{f}'(x) \, \mathrm{e}^{\mathrm{f}(x)} \, \mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + c$
$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln a} + c$	$\int a^{Ax+B} \, \mathrm{d}x = \frac{1}{A} \frac{a^{Ax+B}}{\ln a} + c$	$\int \mathrm{f}'(x) \, a^{\mathrm{f}(x)} \, \mathrm{d}x = \frac{a^{\mathrm{f}(x)}}{\ln a} + c$

Integration of Trigonometric Functions:

$\int \sin x \, \mathrm{d}x = -\cos x + c$	$\int \mathrm{f}'(x) \sin \mathrm{f}(x) \, \mathrm{d}x = -\cos \mathrm{f}(x) + c$
$\int \cos x \, \mathrm{d}x = \sin x + c$	$\int \mathrm{f}'(x) \cos \mathrm{f}(x) \, \mathrm{d}x = \sin \mathrm{f}(x) + c$
$\int \sec^2 x \, \mathrm{d}x = \tan x + c$	$\int \mathrm{f}'(x) \sec^2 \mathrm{f}(x) \, \mathrm{d}x = \tan \mathrm{f}(x) + c$
$\int \operatorname{cosec}^2 x \, \mathrm{d}x = -\cot x + c$	$\int \mathrm{f}'(x) \operatorname{cosec}^2 \mathrm{f}(x) \, \mathrm{d}x = -\cot \mathrm{f}(x) + c$
$\int \sec x \tan x \, \mathrm{d}x = \sec x + c$	$\int \mathrm{f}'(x) \sec \mathrm{f}(x) \tan \mathrm{f}(x) \, \mathrm{d}x = \sec \mathrm{f}(x) + c$
$\int \operatorname{cosec} x \cot x \, \mathrm{d}x = -\operatorname{cosec} x + c$	$\int \mathrm{f}'(x) \operatorname{cosec} \mathrm{f}(x) \cot \mathrm{f}(x) \, \mathrm{d}x = -\operatorname{cosec} \mathrm{f}(x) + c$

Using Trigonometric Identities:

$$\begin{aligned} \int \tan^2 x \, \mathrm{d}x &= \int \sec^2 x - 1 \, \mathrm{d}x \\ &= \tan x - x + c \\ \int \cot^2 x \, \mathrm{d}x &= \int \operatorname{cosec}^2 x - 1 \, \mathrm{d}x \\ &= -\cot x - x + c \end{aligned}$$

Using Double Angle Formulas:

$$\begin{aligned} \int \cos^2 x \, \mathrm{d}x &= \int \frac{1 + \cos 2x}{2} \, \mathrm{d}x \\ &= \frac{x}{2} + \frac{\sin 2x}{4} + c \\ \int \sin^2 x \, \mathrm{d}x &= \int \frac{1 - \cos 2x}{2} \, \mathrm{d}x \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + c \end{aligned}$$

Given in formula list:

$\int \frac{1}{x^2+a^2} \, \mathrm{d}x =$	$\frac{1}{a} \tan^{-1} \frac{x}{a} + c$	
$\int \frac{1}{\sqrt{a^2-x^2}} \, \mathrm{d}x =$	$\sin^{-1} \frac{x}{a} + c$,	$\lvert x \rvert < a$
$\int \frac{1}{x^2-a^2} \, \mathrm{d}x =$	$\frac{1}{2a} \ln \frac{x-a}{x+a} + c$,	$x > a$
$\int \frac{1}{a^2-x^2} \, \mathrm{d}x =$	$\frac{1}{2a} \ln \frac{a+x}{a-x} + c$,	$\lvert x \rvert < a$
$\int \tan x \, \mathrm{d}x =$	$\ln (\sec x) + c$,	$\lvert x \rvert < \frac{\pi}{2}$
$\int \cot x \, \mathrm{d}x =$	$\ln (\sin x) + c$,	$0 < x < \pi$
$\int \operatorname{cosec} x \, \mathrm{d}x =$	$-\ln (\operatorname{cosec} x + \cot x) + c$,	$0 < x < \pi$
$\int \sec x \, \mathrm{d}x =$	$\ln (\sec x + \tan x) + c$,	$\lvert x \rvert < \frac{\pi}{2}$

Integration by Partial Fractions

E.g. $\int \frac{3x^3 + x}{(x+1)^2(x^2+1)}\,dx = \int \frac{3}{x+1} - \frac{2}{(x+1)^2} - \frac{1}{x^2+1}\,dx$ ← split into partial fractions

$$= 3 \ln|x+1| + \frac{2}{x+1} - \tan^{-1} x + c$$

Integration by Splitting the Numerator

E.g. $\int \frac{4x+5}{x^2+2x+3}\,dx$

$= 2\int \frac{2x+2}{x^2+2x+3}\,dx + \int \frac{1}{(x+1)^2+2}\,dx$ ← rewrite the integral as $\int \frac{f'(x)}{f(x)}\,dx + \int \frac{1}{\text{quadratic}}\,dx$

$= 2 \ln|x^2+2x+3| + \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+1}{\sqrt{2}} + c$

Integration by Substitution

E.g. Use the substitution $x = \tan\theta$ to solve (i) $\int \frac{1}{(x^2+1)^{3/2}}\,dx$, (ii) $\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx$.

(i) $x = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$.

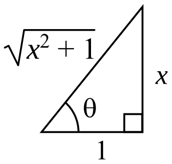
$\int \frac{1}{(x^2+1)^{3/2}}\,dx$

$= \int \frac{1}{(\tan^2\theta+1)^{3/2}} \sec^2\theta\,d\theta$

$= \int \cos\theta\,d\theta$

$= \sin\theta + c$

$= \frac{x}{\sqrt{x^2+1}} + c$



(ii) When $x = 0$, $\theta = 0$.

When $x = 1$, $\theta = \frac{\pi}{4}$.

$\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx = \int_0^{\pi/4} \frac{1}{\sqrt{\tan^2\theta+1}} \sec^2\theta\,d\theta$

$= \int_0^{\pi/4} \sec\theta\,d\theta$

$= [\ln|\sec\theta + \tan\theta|]_0^{\pi/4}$

$= \ln(\sqrt{2} + 1)$

Integration by Parts: $\int uv'\,dx = uv - \int u'v\,dx$

Order for choosing *u*: Logarithm, **I**nverse Trigo, **A**lgebraic, **T**rigonometric, **E**xponential

E.g. $\int_1^e \ln x\,dx$

Let $u = \ln x \Rightarrow u' = \frac{1}{x}$

Let $v' = 1 \Rightarrow v = x$

$= [x \ln x]_1^e - \int_1^e x \frac{1}{x}\,dx$

$= [e \ln e] - \int_1^e 1\,dx$

$= e - [x]_1^e$

$= e - [e - 1] = 1$

E.g. Find $\frac{d}{dx} e^{x^2}$ and deduce $\int x^3 e^{x^2}\,dx$.

$\frac{d}{dx} e^{x^2} = 2xe^{x^2}$

$\int x^3 e^{x^2}\,dx$

$= \frac{1}{2} \int x^2\, 2xe^{x^2}\,dx$ Let $u = x^2$, $v' = 2xe^{x^2}$

$= \frac{1}{2} (x^2 e^{x^2} - \int 2xe^{x^2}\,dx)$

$= \frac{1}{2} (x^2 e^{x^2} - e^{x^2}) + c$

E.g. $\int e^x \sin x\,dx$

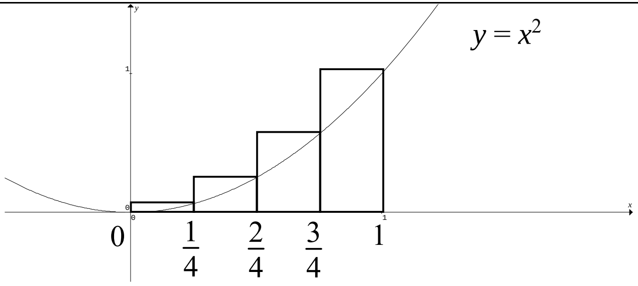
$= e^x \sin x - \int e^x \cos x\,dx$ Let $u = \sin x$, $v' = e^x \Rightarrow u' = \cos x$, $v = e^x$

$= e^x \sin x - [\, e^x \cos x - \int e^x (-\sin x)\,dx \,]$ Let $u = \cos x$, $v' = e^x \Rightarrow u' = -\sin x$, $v = e^x$

$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$

$\int e^x \sin x\,dx = \frac{1}{2} e^x(\sin x - \cos x) + c$

AREA & VOLUME SUMMARY



Area of 4 rectangles

$$= \frac{1}{4}\left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right]$$

Area of n rectangles

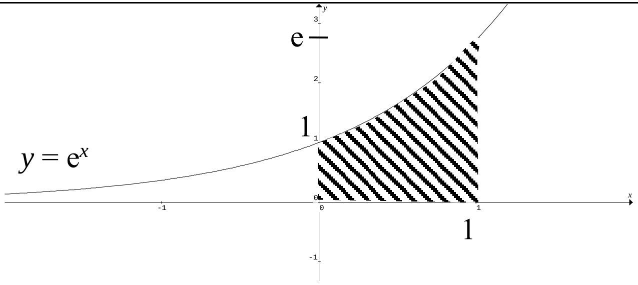
$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \ldots + 1^2\right]$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \ldots + n^2\right]$$

$$= \frac{1}{n^3}\frac{n}{6}(n+1)(2n+1)$$

$$= \frac{1}{6n^2}(2n^2 + 3n + 1)$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \qquad \rightarrow \frac{1}{3} \quad \text{as } n \rightarrow \infty$$



Area $= \int_{\mathbf{0}}^{\mathbf{1}} \mathbf{y\,dx}$

$= \int_0^1 e^x\,dx$

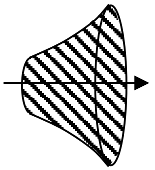
$= [e^x]_0^1$

$= e - 1$

Volume obtained by revolving about x–axis

$= \pi\int_{\mathbf{0}}^{\mathbf{1}} \mathbf{y^2\,dx}$

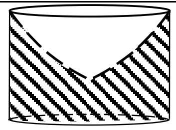
$= \pi\int_0^1 e^{2x}\,dx$

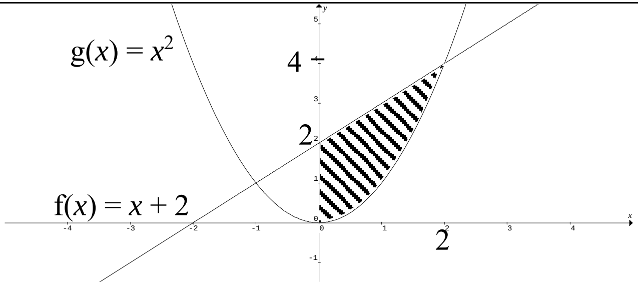


Volume obtained by revolving about y–axis

$= \pi r^2 h - \pi\int_{\mathbf{1}}^{\mathbf{e}} \mathbf{x^2\,dy}$

$= \pi 1^2 e - \pi\int_1^e (\ln x)^2\,dy$





Area $= \int_{\mathbf{0}}^{\mathbf{2}} \mathbf{f(x) - g(x)\,dx}$

$= \int_0^2 (x+2) - x^2\,dx$

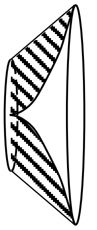
or Area $= \int_0^4 x\,dy \quad -\frac{1}{2}bh$

$= \int_0^4 \sqrt{y}\,dy \quad -\frac{1}{2}(2)(2)$

Volume obtained by revolving about x–axis

$= \pi\int_{\mathbf{0}}^{\mathbf{2}} \mathbf{[\,f(x)\,]^2 - [\,g(x)\,]^2\,dx}$

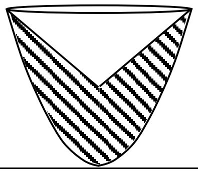
$= \pi\int_0^2 (x+2)^2 - (x^2)^2\,dx$



Volume obtained by revolving about y–axis

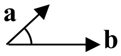
$= \pi\int_0^4 x^2\,dy \quad -\frac{1}{3}\pi r^2 h$

$= \pi\int_0^4 y\,dy \quad -\frac{1}{3}\pi 2^2 2$

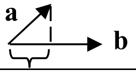


SUMMARY OF VECTORS

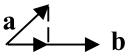
Angle between 2 vectors **a** and **b** $= \cos^{-1} \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$



Length of projection of a vector **a** on a vector **b** $= |\mathbf{a} \bullet \hat{\mathbf{b}}| = \dfrac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{b}|}$



Projection vector of a vector **a** on a vector **b** $= (\mathbf{a} \bullet \hat{\mathbf{b}})\,\hat{\mathbf{b}}$



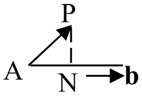
Problems involving a Point and a Line

To check whether a point P lies on a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$:
Equate x, y and z components of P and equation of the line and solve for λ.

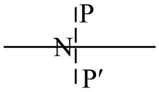
To find the foot of the perpendicular N from a point P to a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$:

Method 1: Use the fact that PN ⊥ **b**, i.e. $(\mathbf{a} + \lambda\mathbf{b} - \mathbf{p}) \bullet \mathbf{b} = 0$ to find λ.

Method 2: $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \overrightarrow{OA}$ + projection of $\overrightarrow{AP}$ on **b** $= \overrightarrow{OA} + (\overrightarrow{AP} \bullet \hat{\mathbf{b}})\hat{\mathbf{b}}$

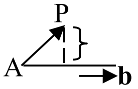


To find the point of reflection P′ of a point P on a line, use $\overrightarrow{ON} = \dfrac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$.



Distance from a point P to a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$: $|\overrightarrow{AP} \times \hat{\mathbf{b}}|$

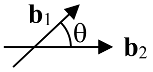
OR if given length of projection of $\overrightarrow{AP}$ on the line, use Pythagoras' Theorem.



Problems involving 2 Lines

To find the point of intersection of 2 lines, equate the x, y and z components.

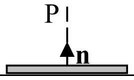
Acute angle between 2 lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ & $\mathbf{r} = \mathbf{a}_2 + \lambda\mathbf{b}_2$: $\cos^{-1} \dfrac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$



Problems involving a Point and a Plane

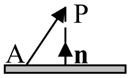
To check whether point P lies on a plane, substitute P into equation of plane.

To find the foot of the perpendicular from a point P to a plane $\mathbf{r} \bullet \mathbf{n} = D$:
Find the intersection of the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$ and the plane.



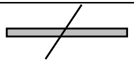
Distance from a point P to a plane $= |\overrightarrow{AP} \bullet \hat{\mathbf{n}}|$, where A is a pt on the plane.

Method 2: Distance from a point P to a plane $\mathbf{r} \bullet \mathbf{n} = D$: $\dfrac{|\mathbf{p} \bullet \mathbf{n} - D|}{|\mathbf{n}|}$



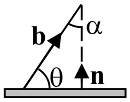
Problems involving a Line and a Plane

To find the intersection of a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane $\mathbf{r} \bullet \mathbf{n} = D$:
Substitute the equation of the line into the equation of the plane.



Angle between a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane $\mathbf{r} \bullet \mathbf{n} = D$:

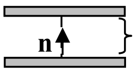
$$\theta = 90° - \cos^{-1} \frac{|\mathbf{b} \bullet \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|} \qquad = \sin^{-1} \frac{|\mathbf{b} \bullet \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$$



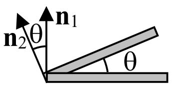
Problems involving 2 Planes

Distance between 2 planes $\mathbf{r} \bullet \mathbf{n} = D_1$ and $\mathbf{r} \bullet \mathbf{n} = D_2$: $\dfrac{|D_1 - D_2|}{|\mathbf{n}|}$

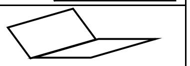
Method 2: Distance $= |\overrightarrow{AB} \bullet \hat{\mathbf{n}}|$, where A, B are 2 points, 1 on each plane.



Angle between 2 planes $\mathbf{r} \bullet \mathbf{n}_1 = D_1$ and $\mathbf{r} \bullet \mathbf{n}_2 = D_2$: $\cos^{-1} \dfrac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$

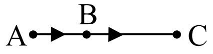


To find the intersection of 2 planes $\mathbf{r} \bullet \mathbf{n}_1 = D_1$ and $\mathbf{r} \bullet \mathbf{n}_2 = D_2$:
Solve the cartesian equations of the 2 planes as simultaneous equations.

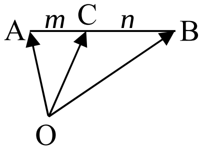


SUMMARY OF VECTORS WITH EXAMPLES

Collinearity: A, B, C are collinear ⇔ $\overrightarrow{AB} = k\overrightarrow{AC}$ for some $k \in \mathbb{R}$.



Ratio Theorem: If C divides AB in the ratio m: n, then $\overrightarrow{OC} = \dfrac{n\overrightarrow{OA} + m\overrightarrow{OB}}{n+m}$.



$\mathbf{a} \bullet \mathbf{b} = \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \cos\theta$

$\mathbf{a} \times \mathbf{b} = \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \sin\theta\, \hat{\mathbf{n}}$, where $\mathbf{n} \perp \mathbf{a}$ & $\mathbf{b}$

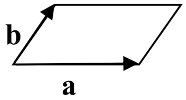
Let **a** and **b** be nonzero vectors. Then $\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a}\bullet\mathbf{b} = 0$

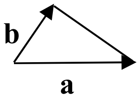
$\mathbf{a} \,/\!/\, \mathbf{b} \Leftrightarrow \mathbf{a}\times\mathbf{b} = \mathbf{0}$

$$\begin{aligned}
(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{a}-\mathbf{b}) &= \mathbf{a}\bullet\mathbf{a} + \mathbf{b}\bullet\mathbf{a} - \mathbf{a}\bullet\mathbf{b} - \mathbf{b}\bullet\mathbf{b} \\
&= \lvert \mathbf{a} \rvert \lvert \mathbf{a} \rvert \cos 0^\circ - \lvert \mathbf{b} \rvert \lvert \mathbf{b} \rvert \cos 0^\circ \\
&= \lvert \mathbf{a} \rvert^2 - \lvert \mathbf{b} \rvert^2 \\
(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b}) &= \mathbf{a}\times\mathbf{a} + \mathbf{b}\times\mathbf{a} - \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{b} \\
&= \mathbf{0} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{a} - \mathbf{0} \\
&= 2\mathbf{b}\times\mathbf{a}
\end{aligned}$$

Area of Parallelogram formed by **a** and **b** = $\lvert \mathbf{a} \times \mathbf{b} \rvert$

Area of Triangle formed by **a** and **b** = $\frac{1}{2}\lvert \mathbf{a} \times \mathbf{b} \rvert$





Equation of a Line:

Cartesian Form: $\dfrac{x-1}{4} = \dfrac{y-2}{5} = \dfrac{z-3}{6}$

Vector Form: $\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\5\\6\end{pmatrix}$, $\lambda \in \mathbb{R}$

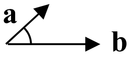
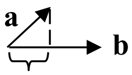
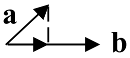
Equation of a Plane:

Cartesian Form: $x + 2y + 3z = 4$

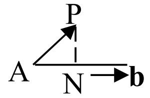
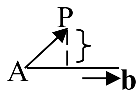
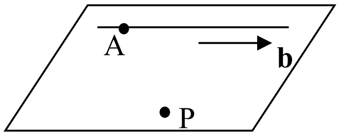
Scalar Product Form: $\mathbf{r} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix} = 4$

Parametric Form: $\mathbf{r} = \begin{pmatrix}0\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-1\end{pmatrix}$ where $\lambda, \mu \in \mathbb{R}$

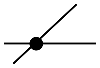
Problems involving 2 Vectors

Angle between 2 vectors **a** and **b** $= \cos^{-1}\dfrac{\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$  	Angle between the 2 vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ $= \cos^{-1}\dfrac{\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}1\\2\\-2\end{pmatrix}}{\sqrt{1^2+2^2+3^2}\sqrt{1^2+2^2+2^2}} = \cos^{-1}\dfrac{-1}{\sqrt{14}\sqrt{9}} = 95.1^\circ$
Length of projection of a vector **a** on another vector **b** = $\lvert \mathbf{a} \bullet \hat{\mathbf{b}} \rvert$  	Length of projection of $\begin{pmatrix}1\\2\\3\end{pmatrix}$ on $\begin{pmatrix}1\\2\\2\end{pmatrix}$ $= \left\lvert \begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \dfrac{1}{\sqrt{9}}\begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert$ $= \dfrac{\lvert 1+4+6 \rvert}{\sqrt{9}} = \dfrac{11}{3}$
Projection vector of **a** on another vector **b** $= (\mathbf{a} \bullet \hat{\mathbf{b}})\,\hat{\mathbf{b}}$  	Projection vector of the vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ on the vector $\begin{pmatrix}1\\2\\2\end{pmatrix}$ $= \left(\begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \dfrac{1}{\sqrt{9}}\begin{pmatrix}1\\2\\2\end{pmatrix}\right)\dfrac{1}{\sqrt{9}}\begin{pmatrix}1\\2\\2\end{pmatrix} \quad = \dfrac{11}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$

Problems involving a Point and a Line

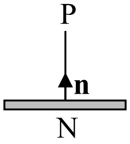
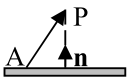
To check whether a point P lies on a line **r** = **a** + λ**b**: Equate the x, y and z components of P and the equation of the line and solve for λ.	Does the point (1, 2, 3) lie on the line $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$? $1 = -1 + \lambda$ ——(1) $2 = 1 + 2\lambda$ ——(2) $3 = 2 + 2\lambda$ ——(3) Equation (1) ⇒ $\lambda = 2$, but equation (2) ⇒ $\lambda = 1/2$ Hence the point (1, 2, 3) does not lie on the line.
To find the foot of the perpendicular N from a point P to a line **r** = **a** + λ**b**:  Use the fact that PN is ⊥ to **b**. **Note:** Reflection P' of P in the line can be found using $\overrightarrow{ON} = \dfrac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$ i.e. $\overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP}$	Find the foot of the perpendicular N from the point P = (2, 2, 4) to the line $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$. $\overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}2\\2\\4\end{pmatrix} = \begin{pmatrix}-3+\lambda\\-1+2\lambda\\-2+2\lambda\end{pmatrix}$ PN ⊥ the line $\Rightarrow \begin{pmatrix}-3+\lambda\\-1+2\lambda\\-2+2\lambda\end{pmatrix} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 0$ $-3 + \lambda - 2 + 4\lambda - 4 + 4\lambda = 0 \Rightarrow \lambda = 1$ $\therefore$ N = (–1 + 1, 1 + 2, 2 + 2) = (0, 3, 4).
Distance from a point P to a line **r** = **a** + λ**b** : $\lvert \overrightarrow{AP} \times \hat{\mathbf{b}} \rvert$ 	Find the distance from P = (1, 2, 3) to $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$. $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$. Distance = $\left\lvert \begin{pmatrix}2\\1\\1\end{pmatrix} \times \frac{1}{\sqrt{9}}\begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert = \frac{1}{3}\left\lvert \begin{pmatrix}0\\-3\\3\end{pmatrix} \right\rvert = \left\lvert \begin{pmatrix}0\\-1\\1\end{pmatrix} \right\rvert = \sqrt{2}$
To find the equation of a plane containing a point P and a line **r** = **a** + λ**b**:  Find normal vector $\mathbf{n} = \overrightarrow{AP} \times \mathbf{b}$. Find constant = $\overrightarrow{OA} \bullet \mathbf{n}$	Find the equation of the plane containing the point P = (1, 2, 3) and the line $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$. $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$. Normal vector = $\begin{pmatrix}2\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}2-2\\-(4-1)\\4-1\end{pmatrix} = \begin{pmatrix}0\\-3\\3\end{pmatrix} = 3\begin{pmatrix}0\\-1\\1\end{pmatrix}$. $\begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}0\\-1\\1\end{pmatrix} = 1 \Rightarrow$ equation of plane is $\mathbf{r} \bullet \begin{pmatrix}0\\-1\\1\end{pmatrix} = 1$.

Problems involving 2 Lines

To check whether 2 lines intersect, or find the point of intersection: 	Do the lines $\mathbf{r} = \begin{pmatrix}0\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ & $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\2\\2\end{pmatrix}$ intersect? $\lambda = -1 + \mu$ ——(1)

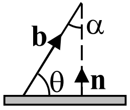
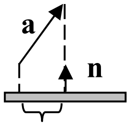
Equate the x, y and z components of the equations of the 2 lines and solve. **Note:** If 2 lines are not parallel and do not intersect, then they are skew.	$3 + 2\lambda = 1 + 2\mu$ ——(2) $3 + 3\lambda = 2 + 2\mu$ ——(3) Solving (2) & (3) gives $\lambda = 1$, $\mu = 2$ Substitute into (1): LHS = 1, RHS = $-1 + 2 = 1$ = LHS. Hence the 2 lines intersect. Point of intersection = $(1, 3 + 2, 3 + 3) = (1, 5, 6)$.
To find the acute angle between 2 lines, use the dot product to find the angle between the 2 direction vectors. $\theta = \cos^{-1}\frac{\lvert \mathbf{b}_1 \bullet \mathbf{b}_2 \rvert}{\lvert \mathbf{b}_1 \rvert \lvert \mathbf{b}_2 \rvert}$	Acute angle between $\mathbf{r} = \begin{pmatrix}0\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\2\\2\end{pmatrix}$ $= \cos^{-1}\frac{\left\lvert \begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert}{\sqrt{1^2+2^2+3^2}\sqrt{1^2+2^2+2^2}} = \cos^{-1}\frac{11}{\sqrt{14}\sqrt{9}} = 11.5^\circ$

Problems involving a Point and a Plane

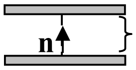
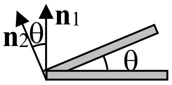
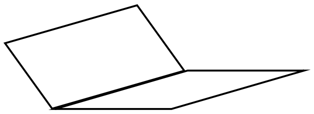
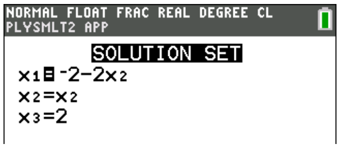
To check whether a point P lies in a plane $\mathbf{r} \bullet \mathbf{n} = D$: Substitute the position vector of P into the equation of the plane.	Does the point (1, 2, 3) lie in the plane $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$? $\begin{pmatrix}1\\2\\3\end{pmatrix} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 1 + 4 + 6 = 11 \neq 2.$ Hence the point (1, 2, 3) does not lie in the plane.
To find the foot of the perpendicular from a point P to a plane $\mathbf{r} \bullet \mathbf{n} = D$:  Find the intersection of the line $\mathbf{r} = \mathbf{p} + \lambda\mathbf{n}$ and the plane. **Note:** Reflection P' of P in the plane can be found using $\overrightarrow{ON} = \frac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$ i.e. $\overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP}$	Find the foot of the perpendicular N from the point P = (1, 2, 3) to the plane $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$. Substitute $\overrightarrow{ON} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$ into the equation $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$: $\begin{pmatrix}1+\lambda\\2+2\lambda\\3+2\lambda\end{pmatrix} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$ $1 + \lambda + 4 + 4\lambda + 6 + 4\lambda = 2$ $9\lambda = -9 \Rightarrow \lambda = -1$ $\therefore N = (1 - 1, 2 - 2, 3 - 2) = (0, 0, 1).$
Distance from a point P to a plane $\mathbf{r} \bullet \mathbf{n} = D$ = length of projection of $\overrightarrow{AP}$ on $\mathbf{n}$.  	Find the distance from (1, 2, 3) to the plane $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$. Pick any point on the plane, e.g. (0, 1, 0). Distance $= \frac{\left\lvert \left[\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}\right] \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert}{\sqrt{1^2+2^2+2^2}} = \frac{\left\lvert \begin{pmatrix}1\\1\\3\end{pmatrix} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert}{3} = \frac{9}{3} = 3$

Problems involving a Line and a Plane

To find the intersection of a line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane $\mathbf{r} \bullet \mathbf{n} = D$:	Find the intersection of $\mathbf{r} = \begin{pmatrix}0\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\3\end{pmatrix}$ and $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$.

Substitute the equation of the line into the equation of the plane.	$\begin{pmatrix}0\\3+2\lambda\\3+3\lambda\end{pmatrix} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$ $6 + 4\lambda + 6 + 6\lambda = 2 \Rightarrow \qquad \lambda = -1$ Point of intersection = (0, 3 – 2, 3 – 3) = (0, 1, 0)
To find the angle between a line **r** = **a** + λ**b** and a plane **r** • **n** = D:  $\theta = 90° - \cos^{-1}\frac{\lvert \mathbf{b}\bullet\mathbf{n} \rvert}{\lvert\mathbf{b}\rvert\,\lvert\mathbf{n}\rvert} = \sin^{-1}\frac{\lvert \mathbf{b}\bullet\mathbf{n} \rvert}{\lvert\mathbf{b}\rvert\,\lvert\mathbf{n}\rvert}$	Angle between the lines $\mathbf{r} = \begin{pmatrix}0\\3\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$ $= \sin^{-1}\frac{\left\lvert \begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert}{\sqrt{1^2+2^2+3^2}\,\sqrt{1^2+2^2+2^2}} = \sin^{-1}\frac{11}{\sqrt{14}\sqrt{9}} = 78.5°$
To find the length of projection of a vector **a** on to a plane **r** • **n** = D:  Use Pythagoras' Theorem	Find the length of projection of $\mathbf{a} = \begin{pmatrix}0\\3\\3\end{pmatrix}$ on $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$. Length of projection of $\begin{pmatrix}0\\3\\3\end{pmatrix}$ on $\begin{pmatrix}1\\2\\2\end{pmatrix} = \left\lvert \begin{pmatrix}0\\3\\3\end{pmatrix} \bullet \frac{1}{\sqrt{9}}\begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert = 4$ $\lvert \mathbf{a} \rvert = \sqrt{0^2+3^2+3^2} = \sqrt{18}$ Length of projection of **a** on the plane = $\sqrt{18-4^2} = \sqrt{2}$. **Method 2:** Length of projection $= \frac{\left\lvert \begin{pmatrix}0\\3\\3\end{pmatrix} \times \begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\left\lvert \begin{pmatrix}0\\3\\-3\end{pmatrix} \right\rvert = \left\lvert \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rvert = \sqrt{2}$

Problems involving 2 Planes

Distance between 2 planes **r** • **n** = D₁ and **r** • **n** = D₂ : $\frac{\lvert D_1 - D_2 \rvert}{\lvert \mathbf{n} \rvert}$  	Distance between the planes $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$ and $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = -4$ $= \frac{\lvert 2-(-4) \rvert}{\sqrt{1^2+2^2+2^2}} = \frac{6}{3} = 2$
Angle between 2 planes **r** • **n**₁ = D₁ and **r** • **n**₂ = D₂: $\cos^{-1}\frac{\lvert \mathbf{n}_1 \bullet \mathbf{n}_2 \rvert}{\lvert \mathbf{n}_1 \rvert\,\lvert \mathbf{n}_2 \rvert}$  	Angle between the planes $\mathbf{r} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix} = 4$ and $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$ $= \cos^{-1}\frac{\left\lvert \begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix} \right\rvert}{\sqrt{1^2+2^2+3^2}\,\sqrt{1^2+2^2+2^2}} = \cos^{-1}\frac{11}{\sqrt{14}\sqrt{9}} = 11.5°$
To find the line of intersection of 2 planes **r** • **n**₁ = D₁ and **r** • **n**₂ = D₂:  Solve the cartesian equations of the 2 planes simultaneously.	Find the line of intersection of $\mathbf{r} \bullet \begin{pmatrix}1\\2\\3\end{pmatrix} = 4$, $\mathbf{r} \bullet \begin{pmatrix}1\\2\\2\end{pmatrix} = 2$. $x + 2y + 3z = 4$ $x + 2y + 2z = 2$ By GC, line of intersection is $\mathbf{r} = \begin{pmatrix}-2\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\0\end{pmatrix}$, $\lambda \in \mathbb{R}$  NORMAL FLOAT FRAC REAL DEGREE CL PLYSMLT2 APP SOLUTION SET x1=-2-2x2 x2=x2 x3=2

SUMMARY OF BINOMIAL EXPANSIONS

$(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + ... + b^n,$

where n is a positive integer, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + ... + \frac{n(n-1)...(n-r+1)}{r!} x^r + ...$ $\quad (| x | < 1)$

Useful Formulas: $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ...$

$(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...$ $\quad (| x | < 1)$

Example: $\frac{9}{(1-x)^2(1+2x)}$

$= \frac{2}{1-x} + \frac{3}{(1-x)^2} + \frac{4}{1+2x}$

$= 2(1-x)^{-1} + 3(1-x)^{-2} + 4(1+2x)^{-1}$

$= 2(1 + x + x^2 + x^3 + ...) + 3[1 - 2(-x) + \frac{-2(-3)}{2}(-x)^2 + \frac{-2(-3)(-4)}{3!}(-x)^3 + ...] + 4[1 - 2x + (2x)^2 - (2x)^3 + ...]$

$= 2 + 2x + 2x^2 + 2x^3 + ... + 3(1 + 2x + 3x^2 + 4x^3 + ...) + 4(1 - 2x + 4x^2 - 8x^3 + ...)$

$= 9 + 27x^2 - 18x^3 + ...$

The expansion is valid for $| x | < 1$ and $| 2x | < 1$

$\Rightarrow | x | < 1$ and $| x | < \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$.

Example: Use the expansion of $\frac{1}{\sqrt{4-x}}$ up to and including the term in x^2 to estimate $\sqrt{2}$.

$\frac{1}{\sqrt{4-x}} = \frac{1}{2\sqrt{1-\frac{x}{4}}}$

$= \frac{1}{2}(1 - \frac{x}{4})^{-1/2}$

$= \frac{1}{2}[1 - \frac{1}{2}(-\frac{x}{4}) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}(-\frac{x}{4})^2 + ...]$

$= \frac{1}{2} + \frac{x}{16} + \frac{3}{256}x^2 + ...$

This expansion is valid for $\left|\frac{x}{4}\right| < 1$, i.e., $-4 < x < 4$.

Put $x = -\frac{1}{2}$: $\frac{1}{\sqrt{9/2}} \approx \frac{1}{2} + \frac{-1/2}{16} + \frac{3}{256}\frac{1}{4}$

$\frac{\sqrt{2}}{3} \approx \frac{483}{1024}$

$\Rightarrow \sqrt{2} \approx \frac{1449}{1024}$

Note: You can also use $x = 2, \frac{7}{2}$, etc., as long as the value is within the range of validity. The approximate values obtained may be different but are acceptable.

SUMMARY OF MACLAURIN'S SERIES

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \qquad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad (\text{all } x)$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

Example: $y = \sqrt{1 + \sin x}$

Method 1 (by differentiation): $y^2 = 1 + \sin x$

$$2y\frac{dy}{dx} = \cos x$$

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = -\sin x$$

$$2y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} + 4\frac{dy}{dx}\frac{d^2y}{dx^2} = -\cos x$$

When $x = 0$, $y = 1$, $\frac{dy}{dx} = \frac{1}{2}$, $\frac{d^2y}{dx^2} = -\frac{1}{4}$, $\frac{d^3y}{dx^3} = -\frac{1}{8}$.

$$\therefore y \quad = 1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2!} - \frac{1}{8}\frac{x^3}{3!} + \ldots \qquad = 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 + \ldots$$

Method 2 (by standard series):

$$y = (1 + \sin x)^{1/2} \qquad = \left[\,1 + \left(x - \frac{x^3}{6} + \ldots\right)\right]^{1/2}$$

$$= 1 + \frac{1}{2}\left(x - \frac{x^3}{6} + \ldots\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(x - \frac{x^3}{6} + \ldots\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(x - \frac{x^3}{6} + \ldots\right)^3 + \ldots$$

$$= 1 + \frac{x}{2} - \frac{1}{2}\frac{x^3}{6} + \ldots - \frac{1}{8}x^2 + \ldots \qquad + \frac{1}{16}x^3 + \ldots$$

$$= 1 + \frac{x}{2} - \frac{1}{8}x^2 - \frac{1}{48}x^3 + \ldots$$

Small Angle Approximation:

If x (in radians) is sufficiently small for x^3 and higher powers of x to be negligible, then

$$\sin x \approx x, \qquad \cos x \approx 1 - \frac{x^2}{2}, \qquad \tan x \approx x.$$

Example: (i) $\dfrac{\cos 2x}{1 - \tan x} \approx \dfrac{1 - \frac{(2x)^2}{2}}{1 - x}$

$$= (1 - 2x^2)(1 - x)^{-1}$$

$$= (1 - 2x^2)(1 + x + x^2 + \ldots)$$

$$= 1 + x + x^2 - 2x^2 + \ldots$$

$$= 1 + x - x^2 + \ldots$$

(ii) $\sin\left(\frac{\pi}{6} + x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x$

$$\approx \frac{1}{2}\left(1 - \frac{x^2}{2}\right) + \frac{\sqrt{3}}{2}x \qquad = \frac{1}{4}\left(2 + 2\sqrt{3}\,x - x^2\right)$$

SUMMARY OF DIFFERENTIAL EQUATIONS

	Example	General Solution	Particular Solution
$\frac{dy}{dx} = f(x)$	$\frac{dy}{dx} = \cos x$	$y = \sin x + C$	If $y = 1$ when $x = 0$, then $C = 1$. So $y = \sin x + 1$
$\frac{dy}{dx} = f(y)$	$\frac{dy}{dx} = \cos^2 y$	$\frac{dx}{dy} = \sec^2 y$ $x = \tan y + C$	If $y = \frac{\pi}{4}$ when $x = 0$, then $C = –1$. So $x = \tan y – 1$
$\frac{dy}{dx} = f(x)g(y)$	$\frac{dy}{dx} = y \cos x$	$\int \frac{1}{y}\, dy = \int \cos x\, dx$ $\ln \mid y \mid = \sin x + C$ $y = \pm e^{\sin x} e^{C} = Ae^{\sin x}$	If $y = 1$ when $x = 0$, then $A = 1$. So $y = e^{\sin x}$

Solving by Substitution:

Use the substitution $u = x + y$ to solve $\frac{dy}{dx} = 2\sqrt{x+y} - 1$.

Given $u = x + y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$

Substitute into D.E.: $\frac{du}{dx} - 1 = 2\sqrt{u} - 1$

$$\frac{du}{dx} = 2\sqrt{u}$$

$$\frac{dx}{du} = \frac{1}{2}u^{-1/2}$$

$$x = \sqrt{u} + C$$

$$x = \sqrt{x+y} + C$$

$$y = (x - C)^2 - x$$

Application of Differential Equations

Water is flowing into a tank at a constant rate, and flowing out at a rate which is proportional to the volume of water in the tank, V m^3. Initially the volume V is 2 m^3 and is decreasing at a rate of 1 m^3/s. When the volume V is 1 m^3, it remains unchanged. Find the volume V in terms of time t s.

$\frac{dV}{dt} = a - bV$, where a, b are constants

When $V = 1$, $\frac{dV}{dt} = 0 \quad \Rightarrow a = b$

$$\Rightarrow \frac{dV}{dt} = a(1 - V)$$

When $t = 0$, $-1 = a(1 - 2) \Rightarrow a = 1$

$$\frac{dt}{dV} = \frac{1}{1 - V}$$

$$t = -\ln \mid 1 - V \mid + C$$

$$\mid 1 - V \mid = e^{C-t}$$

$1 - V = \pm e^{C} e^{-t} = Ae^{-t}$ where $A = \pm e^{C}$

When $t = 0$, $V = 2 \quad \Rightarrow -1 = A$

$$\Rightarrow 1 - V = -e^{-t}$$

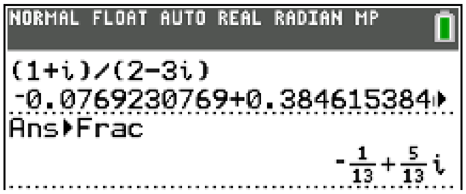
$$\Rightarrow V = 1 + e^{-t}.$$

SUMMARY OF COMPLEX NUMBERS

Examples:

$$\frac{1+\text{i}}{2-3\text{i}}\times\frac{2+3\text{i}}{2+3\text{i}}=\frac{2-3+2\text{i}+3\text{i}}{4+9}$$

$$=-\frac{1}{13}+\frac{5}{13}\,\text{i}$$



NORMAL FLOAT AUTO REAL RADIAN MP

(1+i)/(2-3i)

-0.0769230769+0.384615384▸

Ans▸Frac

$-\frac{1}{13}+\frac{5}{13}i$

Let $(x+\text{i}y)^2 = 3+4\text{i}$ where x, y are real.

$x^2-y^2+2\text{i}xy = 3+4\text{i}$

$x^2-y^2 = 3$ —— (1)

$2xy = 4$ —— (2)

Substitute $y=\frac{2}{x}$ into (1): $x^2-\frac{4}{x^2}=3$

$x^4-3x^2-4=0$

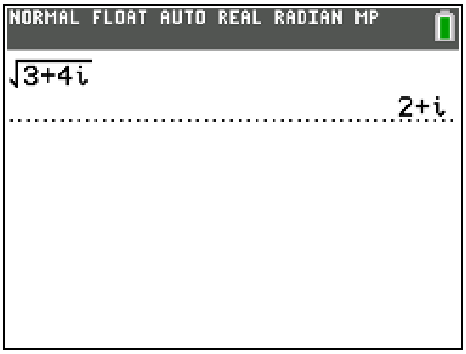
$(x^2-4)(x^2+1)=0$

$x^2 = 4, \ -1$ (rejected)

$x = 2, -2$

$y = 1, -1$

$\therefore x+\text{i}y = 2+\text{i}, -2-\text{i}.$



NORMAL FLOAT AUTO REAL RADIAN MP

$\sqrt{3+4i}$

2+i

$\therefore$ the square roots of 3 + 4i are ±(2 + i).

To solve $2z^2+(2-\text{i})z-\text{i}=0$, z

$$=\frac{-(2-\text{i})\pm\sqrt{(2-\text{i})^2-4(2)(-\text{i})}}{4}$$

$$=\frac{-2+\text{i}\pm\sqrt{3+4\text{i}}}{4}$$

$$=\frac{-2+\text{i}\pm(2+\text{i})}{4}$$

$$=\frac{1}{2}\,\text{i}, -1$$

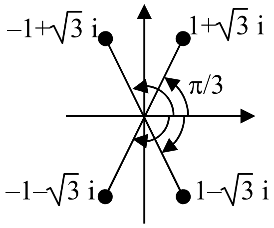
Modulus and Argument

$\arg(1+\sqrt{3}\,\text{i}) = \tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}$

$\arg(1-\sqrt{3}\,\text{i}) = -\tan^{-1}\frac{\sqrt{3}}{1}=-\frac{\pi}{3}$

$\arg(-1+\sqrt{3}\,\text{i}) = \pi-\tan^{-1}\frac{\sqrt{3}}{1}=\frac{2\pi}{3}$

$\arg(-1-\sqrt{3}\,\text{i}) = -(\pi-\tan^{-1}\frac{\sqrt{3}}{1}) = -\frac{2\pi}{3}$



$|z_1 z_2| = |z_1|\,|z_2|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$, $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$

Example: $\left|\frac{-1+\text{i}}{-1-\sqrt{3}\text{i}}\right|$

$$=\frac{|-1+\text{i}|}{|-1-\sqrt{3}\text{i}|}$$

$$=\frac{\sqrt{1^2+1^2}}{\sqrt{1^2+3}}$$

$$=\frac{\sqrt{2}}{2}$$

$$\begin{aligned}
\arg\frac{-1+\mathrm{i}}{-1-\sqrt{3}\mathrm{i}} &= \arg(-1+\mathrm{i}) - \arg(-1-\sqrt{3}\,\mathrm{i}) \\
&= \frac{3\pi}{4} - (-\frac{2\pi}{3}) \\
&= \frac{17\pi}{12} \\
&\equiv \frac{17\pi}{12} - 2\pi \\
&= -\frac{7\pi}{12} \\
|\,(1+\mathrm{i})^{10}\,| &= |\,1+\mathrm{i}\,|^{10} \\
&= \sqrt{2}^{\;10} \\
&= 32 \\
\arg\,(1+\mathrm{i}\,)^{10} &= 10\arg\,(1+\mathrm{i}) \\
&= 10\,(\frac{\pi}{4}) \\
&= \frac{5\pi}{2} \\
&\equiv \frac{5\pi}{2} - 2\pi \\
&= \frac{\pi}{2}
\end{aligned}$$

Example: If $\arg z = \frac{\pi}{3}$ and z^n is real and positive, find the possible integer values of n.

$\arg z^n = \ldots, -2\pi, 0, 2\pi, 4\pi, \ldots$

$\frac{n\pi}{3} = 2k\pi$, where $k \in \mathbb{Z}$

$n = 6k,\ k \in \mathbb{Z}$

If a polynomial with real coefficients has complex roots, then the complex roots occur in conjugate pairs.

Example: Given that $z = 1 + \mathrm{i}$ is a root of the equation $z^3 - z^2 + 2 = 0$, find the other roots.

Since all the coefficients of the equation are real and 1 + i is a root, so 1 – i is also a root.
$[z - (1 + \mathrm{i})]\,[z - (1 - \mathrm{i})] = z^2 - 2z + 2$.
By comparing coefficients, $z^3 - z^2 + 2 = (z^2 - 2z + 2)(z + 1)$.
Hence the other roots are 1 – i and –1.

SUMMARY OF PERMUTATIONS & COMBINATIONS

No. of ways of selecting r objects out of n objects = ^nC_r
No. of ways of arranging n objects = $n!$
No. of ways of arranging n objects with a identical & b identical = $\dfrac{n!}{a!\,b!}$
No. of ways of arranging r objects out of n objects = $n(n-1)(n-2)...(n-(r-1)) = {}^nP_r = {}^nC_r\, r!$
No. of ways of arranging n objects in a circle = $(n-1)!$

Choosing People

A team of 4 is to be chosen from a group consisting of Anne and Bob and 4 other people. In how many ways can this be done if

(i) there are no restrictions?
(ii) Anne must be in the team?
(iii) Anne and Bob must both be in the team?
(iv) at most one of Anne and Bob in the team?
(v) Anne or Bob or both are in the team?

(i) No. of ways of choosing 4 people = $^6C_4 = 15$
(ii) No. of ways of choosing the other 3 people = $^5C_3 = 10$
(iii) No. of ways of choosing the other 2 people = $^4C_2 = 6$
(iv) Total no. of ways – no. of ways with Anne & Bob both in the team = 15 – 6 = 9
(v) No. of ways with Anne in the team + no. of ways with Bob in the team – no. of ways with both in the team = 10 + 10 – 6 = 14

Choosing from Different Types of People (e.g. Boys & Girls)

In how many ways can a team of 3 be chosen from a group of 3 boys and 4 girls if

(i) there are no restrictions?
(ii) there must be exactly 1 boy?
(iii) there must be at least 1 boy?
(iv) there must be at least 1 boy and at least 1 girl?

(i) No. of ways of choosing 4 people = $^7C_3 = 35$
(ii) No. of ways of choosing 1 boy and 2 girls = $^3C_1\, {}^4C_2 = 18$
(iii) No. of ways = Total no. of ways – no. of ways with no boys = $35 - {}^4C_3 = 35 - 4 = 31$
Note: It is wrong to say no. of ways = $^3C_1\, {}^6C_2 = 45$
(iv) Total no. of ways – no. of ways with no boys or no girls = $35 - {}^4C_3 - {}^3C_3 = 30$
Note: It is wrong to say no. of ways = $^3C_1\, {}^4C_1\, {}^5C_1 = 60$

Choosing People to form Groups

Find the no. of ways in which 6 people can be divided into

(i) two groups consisting of 4 & 2 people,
(ii) two groups consisting of 3 people each.
(iii) group A and group B, with 3 people in each group.

(i) No. of ways = $^6C_4\, {}^2C_2 = 15$
(ii) No. of ways = $\dfrac{^6C_3\, {}^3C_3}{2!} = 10$ **Note:** Divide by 2! since the 2 groups are of equal size.
(iii) No. of ways = $^6C_3\, {}^3C_3 = 20$ **Note:** Don't divide by 2! since the 2 groups are labeled.

Choosing Letters, Balls or other Identical Objects

Find the number of ways three cards can be selected from seven cards which together spell the word "MINIMUM". The order of selection is not important.

Case 1: No. of ways of choosing 3 identical letters = no. of ways of choosing "MMM" = 1
Case 2: No. of ways of choosing "MM" + 1 other letter
= no. of ways of choosing I, N, U = $^3C_1 = 3$
Case 3: No. of ways of choosing "I I" + 1 other letter
= no. of ways of choosing M, N, U = $^3C_1 = 3$
Case 4: No. of ways of choosing 3 different letters
= no. of ways of choosing M, I, N, U = $^4C_3 = 4$
$\therefore$ total no. of selections = 1 + 3 + 3 + 4 = 11

Arranging People
Anne and Bob and 2 other people are to sit in a row. How many ways can this be done if
(i) there are no restrictions?
(ii) Anne must sit on the left and Bob on the right?
(iii) Anne and Bob must sit together?
(iv) Anne and Bob must be separate?

(i) No. of ways of arranging 4 people = 4! = 24
(ii) No. of ways of arranging the other 2 people = 2! = 2
(iii) Treat Anne and Bob as 1 item.
No. of ways of arranging 3 items × no. of ways of arranging Anne & Bob = 3! 2! = 12
(iv) No. of ways = 24 – 12 = 12

Arranging Different Types of People (e.g. Boys & Girls)
In how many ways can 3 boys & 3 girls be arranged in a row if
(i) there are no restrictions?
(ii) the 1st person on the left is a boy?
(iii) the person on each end is a boy?
(iv) the boys are together?
(v) the boys are separate?

(i) No. of ways of arranging 6 people = 6! = 720
(ii) No. of arrangements = $^3C_1$ × no. of ways of arranging other 5 people = $^3C_1\ 5! = 360$
(iii) No. of arrangements = $^3C_2 \times 2!$ × no. of ways of arranging the other 4 people
= $^3C_2\ 2!\ 4! = 144$
(iv) Treat the 3 boys as 1 item. No. of ways of arranging 4 items = 4!
The boys can be arranged among themselves in 3! ways
$\therefore$ no. of arrangements = 4! 3! = 144
(v) G_1 G_2 G_3
↑ ↑ ↑ ↑
The 3 girls can be arranged in 3! ways.
From the 4 spaces, choose 3 places for the boys in $^4C_3$ ways, and arrange them in 3! ways.
$\therefore$ no. of arrangements = $3!\ ^4C_3\ 3! = 144$
Note: It is wrong to subtract no. of ways where boys are together from the total no. of ways.

Arranging Letters, Balls or other Identical Objects
(a) Find the number of arrangements of all 7 letters of the word "MINIMUM" in which
  (i) there are no restrictions.
  (ii) the 3 letters M are next to each other
  (iii) the 3 letters M are separate
  (iv) the first letter is M
  (v) the first & last letters are M

(vi) the first letter is M or the last letter is M or both

(b) Find the number of 4–letter code–words that can be made from the letters of the word "MINIMUM".

(ai) No. of arrangements = $\frac{7!}{3!\,2!}$ = 420

(ii) Treat "MMM" as 1 item.

No. of arrangements of "MMM", I, N, I, U = $\frac{5!}{2!}$ = 60

(iii) I N I U

↑ ↑ ↑ ↑ ↑

The letters I, N, I, U can be arranged in $\frac{4!}{2!}$ ways

From the 5 spaces, choose 3 places for the 3 M's in ${}^5C_3$ ways.

$\therefore$ no. of arrangements = $\frac{4!}{2!}\,{}^5C_3$ = 120

(iv) M _ _ _ _ _ _

No. of ways of arranging I, N, I, M, U, M = $\frac{6!}{2!\,2!}$ = 180

(v) M _ _ _ _ _ M

No. of ways of arranging I, N, I, M, U = $\frac{5!}{2!}$ = 60

(vi) No. of arrangements = 180 + 180 – 60 = 300

(b) Case 1: M, M, M & 1 other letter: No. of words = ${}^3C_1\,\frac{4!}{3!}$ = 12

Case 2: M, M, I, I: No. of words = $\frac{4!}{2!\,2!}$ = 6

Case 3: M, M & 2 different letters: No. of words = ${}^3C_2\,\frac{4!}{2!}$ = 36

Case 4: I, I & 2 different letters: No. of words = ${}^3C_2\,\frac{4!}{2!}$ = 36

Case 5: 4 different letters: No. of words = 4! = 24

Total no. of code–words = 12 + 6 + 36 + 36 + 24 = 114

Arranging People Around a Table

4 men and 3 women are to sit at a round table. Find the number of ways of arranging them if

(i) there are no restrictions.

(ii) the 3 women must sit together.

(iii) the 3 women must be separate.

(iv) the 3 women must be separate and the seats are numbered 1 to 7.

(i) No. of ways of arranging 7 people around a table = (7 – 1)! = 720

(ii) Treat the 3 women as 1 item. Arrange 5 items around a table in (5 – 1)! ways.

The 3 women can be arranged among themselves in 3! ways

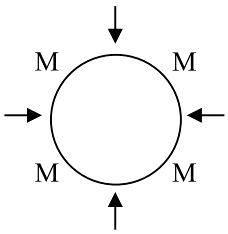
$\therefore$ no. of arrangements = (5 – 1)! 3! = 144

(iii) Let the 4 men sit down first in (4 – 1)! ways.

From the 4 spaces, choose 3 places for the women in ${}^4C_3$ ways.

Arrange the 3 women in 3! ways.

$\therefore$ total no. of arrangements = (4 – 1)! ${}^4C_3$ 3! = 144



(iv) Since the seats are numbered, no. of arrangements = 144 × 7 = 1008

SUMMARY OF PROBABILITY

Formulas:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A) = P(A \cap B) + P(A \cap B')$

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

If A & B are independent, then $P(A \cap B) = P(A)P(B)$, $P(A \mid B) = P(A)$, $P(B \mid A) = P(B)$.

If A & B are mutually exclusive, then $P(A \cap B) = 0$.

Events A and B are such that $P(A \mid B) = \frac{1}{4}$, $P(B) = \frac{1}{2}$, $P(A \cap B') = \frac{1}{8}$, find

(i) P(A). Are events A and B mutually exclusive? Are they independent?
(ii) $P(B \mid A)$
(iii) $P(A \cup B)$
(iv) $P(A' \cap B')$

(i) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A \cap B) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

$\therefore P(A) = P(A \cap B) + P(A \cap B') = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

Since $P(A \cap B) = \frac{1}{8} \neq 0$, events A and B are not mutually exclusive.

Since $P(A \mid B) = \frac{1}{4} = P(A)$, events A & B are independent.

Method 2: Since $P(A)\,P(B) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} = P(A \cap B)$, events A & B are independent.

(ii) $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/8}{1/4} = \dfrac{1}{2}$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$

(iv) $P(A' \cap B') = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$

Throw two fair dice. Find the probability that
(i) at least one of the scores is 6,
(ii) the sum of the two scores is > 9,
(iii) the sum of the two scores is > 9 and at least one of the scores is 6,
(iv) the sum of the two scores is > 9, given that at least one of the scores is 6.
Are the events in (i) and (ii) independent?

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(i) P(at least one of the scores is 6) $= \frac{11}{36}$

Method 2: 1 – P(both scores ≤ 5) $= 1 - \frac{5}{6}\frac{5}{6} \quad = \frac{11}{36}$

(ii) P(the sum of the 2 scores is > 9) $= \frac{6}{36} \quad = \frac{1}{6}$

(iii) P(sum > 9 and at least one of the scores is 6)

= P(‘4, 6’, ‘5, 6’, ‘6, 4’, ‘6, 5’, ‘6, 6’) $= \frac{5}{36}$

(iv) P(sum > 9 | at least one of the scores is 6)

$$= \frac{\text{P(sum} > 9 \cap \text{at least one of the scores is 6)}}{\text{P(at least one of the scores is 6)}} \qquad = \frac{5/36}{11/36} \qquad = \frac{5}{11}$$

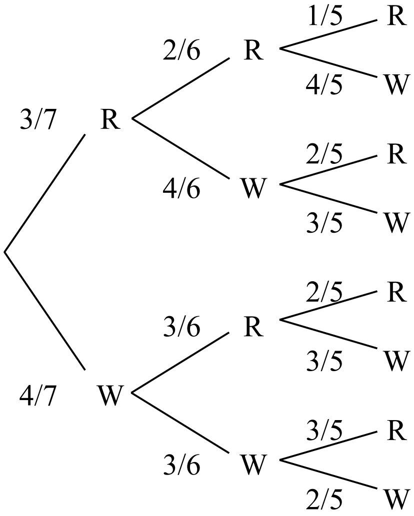
Since P(sum > 9 | ≥ one score is 6) $= \frac{5}{11} \neq \frac{1}{6}$ = P(sum > 9), the 2 events are not independent.

A bag contains 3 red balls & 4 white balls. 3 balls are drawn from the bag at random and **without replacement**. Find the probability that

(i) 2 of the balls are red,

(ii) at least 1 of the balls is red,

(iii) 2 of the balls are red, given that at least 1 of the balls is red.



(i) P(RRW, RWR, WRR) $= \frac{3}{7}\frac{2}{6}\frac{4}{5} + \frac{3}{7}\frac{4}{6}\frac{2}{5} + \frac{4}{7}\frac{3}{6}\frac{2}{5} = \frac{12}{35}$

Method 2: P(2 red) $= \frac{{}^3C_2\ {}^4C_1}{{}^7C_3} = \frac{3 \times 4}{35} = \frac{12}{35}$

(ii) P(at least 1 red) = 1 – P(WWW)

$= 1 - \frac{4}{7}\frac{3}{6}\frac{2}{5} = \frac{31}{35}$

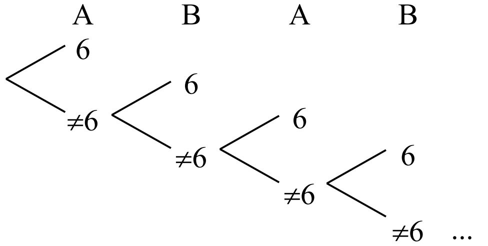
Method 2: P(at least 1 red) = 1 – P(no red)

$= 1 - \frac{{}^4C_3}{{}^7C_3} = 1 - \frac{4}{35} = \frac{31}{35}$

(iii) P(2 red | at least 1 red) $= \frac{\text{P(2 red} \cap \text{at least 1 red)}}{\text{P(at least 1 red)}}$

$= \frac{12/35}{31/35} = \frac{12}{31}$

In a game, Adam and Bob take turns at rolling a fair die, starting with Adam. The first person to obtain a “6” wins the game. Find the probability that
(i) Adam wins the game on his second roll;
(ii) the winner wins the game on his second roll;
(iii) Adam wins the game.



(i) P(Adam wins on 2nd roll) = $\frac{5}{6}\frac{5}{6}\frac{1}{6} = \frac{25}{216}$

(ii) P(The winner wins on 2nd roll)

= P(Adam wins on 2nd roll) + P(Bob wins on 2nd roll) = $\frac{25}{216} + \frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{1}{6} = 0.212$

(iii) P(Adam wins) = $\frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{1}{6} + \ldots \quad = \frac{1/6}{1 - \frac{25}{36}} \quad = \frac{6}{11}$

There are 11 people at a gathering. There are 3 couples, 2 single men and 3 single women.
(a) If 4 people are chosen at random from the gathering, find the probability that there are
  (i) 2 men and 2 women,
  (ii) at least 3 women.
(b) If 2 people are chosen at random from the gathering, find the probability that they are
  (i) both women,
  (ii) both single, given that they are both women,
  (iii) married to each other,
  (iv) a man and a woman.

(ai) P(2 men and 2 women) = $\frac{{}^{5}C_{2}\,{}^{6}C_{2}}{{}^{11}C_{4}} = \frac{5}{11}$

Method 2: P(2 men and 2 women) = $\frac{5}{11}\frac{4}{10}\frac{6}{9}\frac{5}{8}\frac{4!}{2!\,2!} = \frac{5}{11}$

(ii) P(at least 3 women) = P(exactly 3 women) + P(4 women) $= \frac{{}^{5}C_{1}\,{}^{6}C_{3}}{{}^{11}C_{4}} + \frac{{}^{6}C_{4}}{{}^{11}C_{4}} = \frac{23}{66}$

(bi) P(both women) = $\frac{{}^{6}C_{2}}{{}^{11}C_{2}} = \frac{15}{55} = \frac{3}{11}$ **Method 2:** P(both women) = $\frac{6}{11}\frac{5}{10} = \frac{3}{11}$

(ii) P(both single | both women) = $\frac{\text{P(both women and single)}}{\text{P(both women)}} = \frac{\frac{{}^{3}C_{2}}{{}^{11}C_{2}}}{3/11} = \frac{3/55}{3/11} = \frac{1}{5}$

(iii) P(married to each other) = $\frac{{}^{3}C_{1}}{{}^{11}C_{2}} = \frac{3}{55}$

Method 2: P(married to each other) = $\frac{6}{11}\frac{1}{10} = \frac{3}{55}$

(iv) P(a man and a woman) = $\frac{{}^{5}C_{1}\,{}^{6}C_{1}}{{}^{11}C_{2}} = \frac{5 \times 6}{55} = \frac{6}{11}$

Method 2: P(a man and a woman) = $\frac{5}{11}\frac{6}{10} \times 2! = \frac{6}{11}$

SUMMARY OF DISCRETE RANDOM VARIABLES

Mean $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Variance $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{n}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$

Discrete Random Variables $0 \le \mathrm{P}(X = x) \le 1$, $\sum_{\text{all } x} \mathrm{P}(X = x) = 1$

Expectation or Expected Value E(X) $= \sum_{\text{all } x} x\mathrm{P}(X = x)$

$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$

$\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$

$\mathrm{E}[\mathrm{g}(X)] = \sum_{\text{all } x} \mathrm{g}(x)\mathrm{P}(X = x)$

Variance Var(X) $= \mathrm{E}[(X - \mu)^2] = \mathrm{E}(X^2) - [\ \mathrm{E}(X)\ ]^2$

$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

If X and Y are independent, then $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$.

Example: A random variable X has the following probability distribution:

x	1	2	3	4
$\mathrm{P}(X = x)$	0.1	0.2	0.3	k

Find (i) k, (ii) $\mathrm{P}(X \le 2)$, (iii) $\mathrm{E}(X)$, (iv) $\mathrm{Var}(X)$.

(i) $\sum_{\text{all } x} x\mathrm{P}(X = x) = 0.1 + 0.2 + 0.3 + k = 1$

$\Rightarrow k = 0.4$

(ii) $\mathrm{P}(X \le 2) = \mathrm{P}(X = 1) + \mathrm{P}(X = 2)$

$= 0.1 + 0.2$

$= 0.3$

(iii) $\mathrm{E}(X) = 1(0.1) + 2(0.2) + 3(0.3) + 4(0.4)$

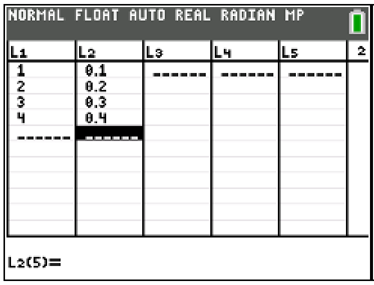
$= 3$

(iv) $\mathrm{E}(X^2) = 1^2(0.1) + 2^2(0.2) + 3^2(0.3) + 4^2(0.4)$

$= 10$

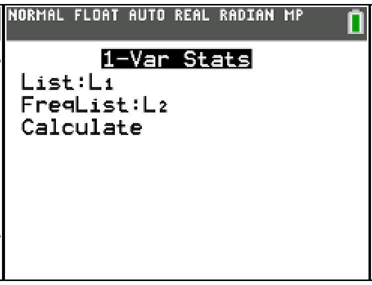
$\Rightarrow \mathrm{Var}(X) = 10 - 3^2 = 1$

Method 2:

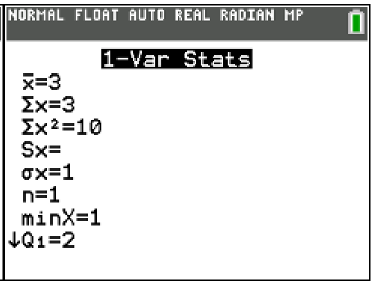


```
L1     L2     L3     L4     L5     2
1      0.1    ------ ------ ------
2      0.2
3      0.3
4      0.4
------ ------

L2(5)=
```



```
1-Var Stats
List:L1
FreqList:L2
Calculate
```



```
1-Var Stats
x̄=3
Σx=3
Σx²=10
Sx=
σx=1
n=1
minX=1
↓Q1=2
```

By GC, $\mathrm{E}(X) = 3$,

$\mathrm{Var}(X) = 1^2 = 1$.

BINOMIAL DISTRIBUTION $X \sim \mathrm{B}(n, p)$

Characteristics:

(i) The experiment consists of n repeated independent trials.
(ii) Each trial has two possible outcomes: a ‘success’ or a ‘failure’.
(iii) The probability of a success is constant.

$\mathrm{P}(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$

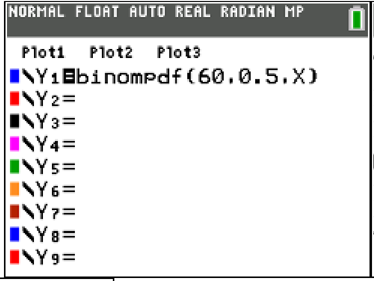
$\mathrm{E}(X) = np$

$\mathrm{Var}(X) = np(1 - p)$

In an experiment, a fair coin is tossed 60 times.
(i) Find the probability that there are exactly 25 heads.
(ii) Find the probability that there are at most 25 heads.
(iii) Find the probability that there are at least 25 heads.
(iv) Find the most likely number of heads obtained.
(v) If a fair coin is tossed n times, find the least value of n such that the probability of getting at least 2 heads exceeds 0.99.

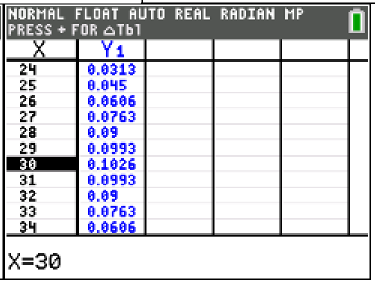
Let X = no. of heads out of 60 tosses. $X \sim \mathrm{B}(60, 0.5)$.

(i) $\mathrm{P}(X = 25) = 0.0450$ ← binomialpdf(60, 0.5, 25)
(ii) $\mathrm{P}(X \leq 25) = 0.123$ ← binomialcdf(60, 0.5, 25)
(iii) $\mathrm{P}(X \geq 25) = 1 - \mathrm{P}(X \leq 24) = 0.922$ ← 1 – binomialcdf(60, 0.5, 24)
(iv)



```
NORMAL FLOAT AUTO REAL RADIAN MP

 Plot1  Plot2  Plot3
\Y1=binompdf(60,0.5,X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl
  X     Y1
  24    0.0313
  25    0.045
  26    0.0606
  27    0.0763
  28    0.09
  29    0.0993
  30    0.1026
  31    0.0993
  32    0.09
  33    0.0763
  34    0.0606
X=30
```

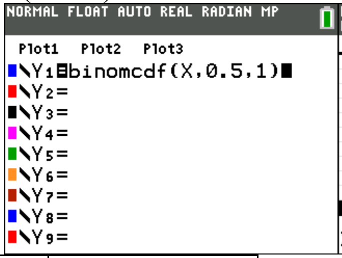
x	$\mathrm{P}(X = x)$
29	0.0993
30	0.1026
31	0.0993

By GC, most likely number of heads = 30.

(v) Let W = no. of heads out of n tosses. $W \sim \mathrm{B}(n, 0.5)$.

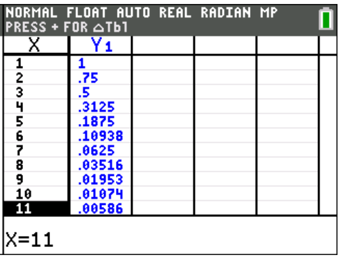
$\mathrm{P}(W \geq 2) > 0.99$

$\mathrm{P}(W \leq 1) < 0.01$



```
NORMAL FLOAT AUTO REAL RADIAN MP

 Plot1  Plot2  Plot3
\Y1=binomcdf(X,0.5,1)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl
  X     Y1
  1     1
  2     .75
  3     .5
  4     .3125
  5     .1875
  6     .10938
  7     .0625
  8     .03516
  9     .01953
  10    .01074
  11    .00586
X=11
```

n	$\mathrm{P}(W \leq 1)$
10	$0.0107 > 0.01$
11	$0.00586 < 0.01$

$\therefore$ least $n = 11$

SUMMARY OF NORMAL DISTRIBUTION $X \sim N(\mu , \sigma^2)$

$P(X < a) = P(Z < \frac{a-\mu}{\sigma})$, where $Z \sim N(0, 1)$.

If $X_1 , X_2 , ..., X_n \sim N(\mu , \sigma^2)$, then $X_1 + X_2 + ... + X_n \sim N(n\mu, n\sigma^2)$.

If $X \sim N(\mu , \sigma^2)$, then $nX \sim N(n\mu, n^2\sigma^2)$ and $\overline{X} \sim N(\mu , \frac{\sigma^2}{n})$.

If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ are independent, then $aX + bY \sim N(a\mu_1+b\mu_2, a^2\sigma_1^2+b^2\sigma_2^2)$

The mass of melons and pumpkins have distributions $N(\mu, \sigma^2)$ and $N(3, 1)$ respectively. Given that 2.275% of melons weigh < 1 kg and 15.8655% weigh > 2.5 kg, find μ and σ.
Find the probability that
(i) 5 melons weigh less than 12 kg altogether,
(ii) the mean mass of 5 melons is less than 2.4 kg,
(iii) in a sample of 5 melons, at most 1 melon weigh less than 1 kg,
(iv) 4 melons weigh less than 2 pumpkins,
(v) 4 melons weigh less than twice the mass of 1 pumpkin.

Let M = mass of a melon in kg.

$P(M < 1) = 2.275\%$	$P(M > 2.5) = 15.8655\%$
$P(Z < \frac{1-\mu}{\sigma}) = 0.02275$	$P(Z > \frac{2.5-\mu}{\sigma}) = 0.158655$
$\frac{1-\mu}{\sigma} = -2.00000$	$\frac{2.5-\mu}{\sigma} = 1.00000$
$1 - \mu = -2\sigma$ —— (1)	$2.5 - \mu = \sigma$ —— (2)

$(2) - (1) \Rightarrow 3\sigma = 1.5 \Rightarrow \sigma = 0.5, \mu = 2$

(i) $M_1 + M_2 + ... + M_5 \sim N(5\times2, 5\times0.5^2) = N(10, 1.25)$
$P(M_1 + M_2 + ... + M_5 < 12) = 0.963 \quad \leftarrow \boxed{\text{normalcdf}(-10^{99}, 10.5, 10, \sqrt{1.25})}$

(ii) $\overline{M} \sim N(2, \frac{0.5^2}{5}) = N(2, 0.05)$
$P(\overline{M} < 2.4) = 0.963 \quad \leftarrow \boxed{\text{normalcdf}(-10^{99}, 2.4, 2, \sqrt{0.05})}$

(iii) Let X = no. of melons out of 5 which weigh < 1 kg.
$X \sim B(5, 0.02275)$
$P(X \le 1) = 0.995 \quad \leftarrow \boxed{\text{binomialcdf}(5, 0.02275, 1)}$

(iv) Let P = mass of a pumpkin in kg.
$M_1 + M_2 + M_3 + M_4 - P_1 - P_2 \sim N(4\times2 - 2\times3, 4\times0.5^2 + 2\times1^2) = N(2, 3)$
$P(M_1 + M_2 + M_3 + M_4 < P_1 + P_2)$
$= P(M_1 + M_2 + M_3 + M_4 - P_1 - P_2 < 0)$
$= 0.124 \quad \leftarrow \boxed{\text{normalcdf}(-10^{99}, 0, 2, \sqrt{3})}$

(v) $M_1 + M_2 + M_3 + M_4 - 2P \sim N(4\times2 - 2\times3, 4\times0.5^2 + 2^2\times1^2) = N(2, 5)$
$P(M_1 + M_2 + M_3 + M_4 < 2P)$
$= P(M_1 + M_2 + M_3 + M_4 - 2P < 0)$
$= 0.186 \quad \leftarrow \boxed{\text{normalcdf}(-10^{99}, 0, 2, \sqrt{5})}$

SUMMARY OF SAMPLING

A sample for which every individual of the population has an **equal chance of being selected** and individuals are **selected independently** of one another is called a **random sample**.

Unbiased estimate of population variance σ^2: $s^2 = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right) = \frac{n}{n-1}\left(\frac{\Sigma(x-\bar{x})^2}{n}\right)$

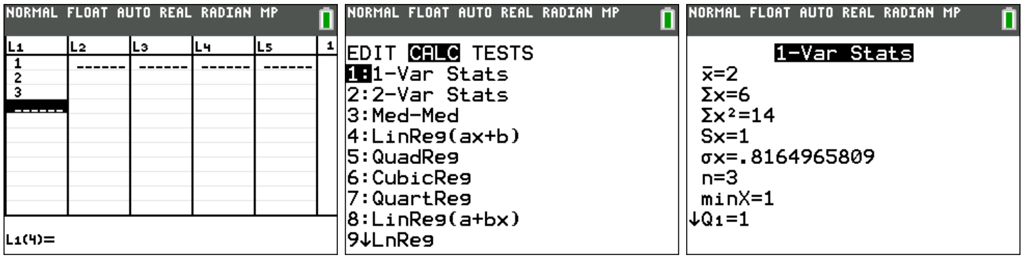
If $X \sim N(\mu, \sigma^2)$, then $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ and $X_1 + ... + X_n \sim N(n\mu, n\sigma^2)$.

Central Limit Theorem:

If n large (X not necessarily normal), then $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ and $X_1 + ... + X_n \sim N(n\mu, n\sigma^2)$ approx.

Find unbiased estimates of the population mean and variance given the sample data 1, 2, 3.

Method 1:



```
NORMAL FLOAT AUTO REAL RADIAN MP
L1    L2     L3     L4     L5    1
1     ------ ------ ------ ------
2
3
------
L1(4)=
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)
9↓LnReg
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
      1-Var Stats
 x̄=2
 Σx=6
 Σx²=14
 Sx=1
 σx=.8164965809
 n=3
 minX=1
↓Q1=1
```

Unbiased estimate of population mean $\mu = \bar{x} = 2$.

Unbiased estimate of population variance $\sigma^2 = s^2 = 1^2 = 1$.

Method 2: $\bar{x} = \frac{1+2+3}{3} = 2$

$$s^2 = \frac{(1-2)^2 + (2-2)^2 + (3-2)^2}{2} = 1.$$

Find unbiased estimates of the population mean and variance given sample size $n = 10$ and

(i) $\Sigma x = 50$, $\Sigma x^2 = 400$

(ii) $\Sigma x = 50$, $\Sigma(x - \bar{x})^2 = 400$

(iii) $\Sigma(x - 3) = 50$, $\Sigma(x - 3)^2 = 400$

(iv) $\Sigma x = 50$, sample variance = 45

(i) $\bar{x} = \frac{50}{10} = 5$, $s^2 = \frac{1}{9}\left(400 - \frac{50^2}{10}\right) = \frac{50}{3}$

(ii) $\bar{x} = \frac{50}{10} = 5$, $s^2 = \frac{400}{9}$

(iii) $\bar{x} = \frac{50}{10} + 3 = 8$, $s^2 = \frac{1}{9}\left(400 - \frac{50^2}{10}\right) = \frac{50}{3}$

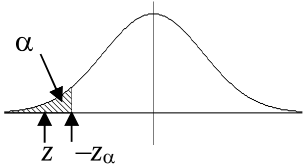
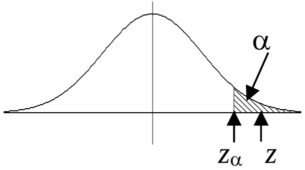
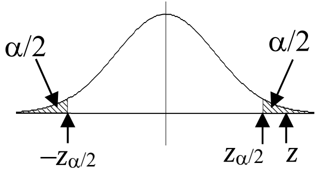
(iv) $\bar{x} = \frac{50}{10} = 5$, $s^2 = \frac{10}{9} \times 45 = 50$

A random variable X has a distribution with mean 20 and variance 10. Find the probability that the mean of a random sample of 50 values of X exceeds 21.

Since $n = 50$ is large, by CLT, $\overline{X} \sim N(20, \frac{10}{50}) = N(20, 0.2)$ approximately.

$P(\overline{X} > 21) = 0.0127$ ← normalcdf(21, 10^{99}, 20, $\sqrt{0.2}$)

SUMMARY OF HYPOTHESIS TESTING

$H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$	$H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$	$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$
p-value = $P(Z < \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}})$ 	p-value = $P(Z > \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}})$ 	p-value = $P(\lvert Z \rvert > \frac{\lvert \overline{x} - \mu_0 \rvert}{\sigma / \sqrt{n}})$ 

Reject H_0 if p-value ≤ α

Do not reject H_0 if p-value > α

Example (1-tailed test):

A machine was calibrated to manufacture 1-metre rods. The lengths x of a sample of 100 rods were recorded. It was found that $\Sigma x = 90$ and $\Sigma x^2 = 92$.

(i) Test, at the 5% level, whether the rods produced by the machine are too short.

(ii) Explain the meaning of "5% level of significance" and "p-value".

(iii) Another sample of 100 rods was tested and the null hypothesis was rejected at 5% level of significance. Using the same value of s^2 found above, find the range of possible values of $\overline{x}$.

(iv) Suppose $H_0 : \mu = 1$ was tested against $H_1 : \mu \neq 1$ at 10% level of significance. Without carrying out any further tests, decide whether the same conclusion would be obtained.

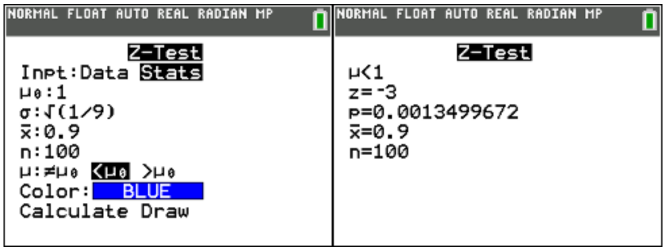
$\overline{x} = \frac{90}{100} = 0.9$, $\quad s^2 = \frac{1}{99}[92 - \frac{90^2}{100}] = \frac{1}{9}$

(i) $H_0 : \mu = 1$

$H_1 : \mu < 1$

Under H_0 , since $n = 100$ is large, by CLT, $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ approximately.

Test statistic $Z = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} \sim N(0, 1)$.



```
NORMAL FLOAT AUTO REAL RADIAN MP
Z-Test
Inpt:Data Stats
μ0:1
σ:√(1/9)
x̄:0.9
n:100
μ:≠μ0 <μ0 >μ0
Color: BLUE
Calculate Draw
```

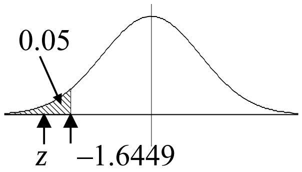
```
NORMAL FLOAT AUTO REAL RADIAN MP
Z-Test
μ<1
z=-3
p=0.0013499672
x̄=0.9
n=100
```

Since p-value = 0.00135 < 0.05, we reject H_0. There is sufficient evidence at the 5% level to say that the rods produced by the machine are too short.

(ii) "5% level of significance" means that the probability of wrongly concluding that the population mean length is < 1 m, when the population mean length is actually 1 m, is 0.05.

The "p-value" is the probability of getting a sample mean length ≤ 0.9 m, when the population mean length is 1 m.

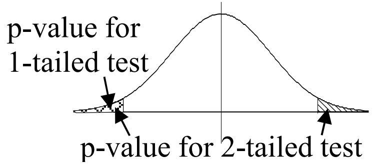
(iii) H_0 is rejected $\Rightarrow \frac{\overline{x} - 1}{\sqrt{\frac{1/9}{100}}} \leq -1.6449$



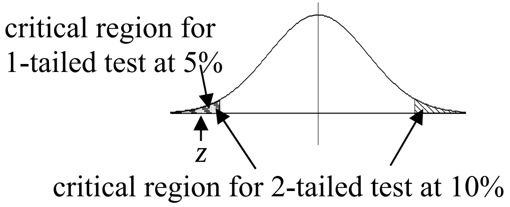
$\overline{x} - 1 \leq -0.054828$

$\overline{x} \leq 0.945$

(iv) We explain using either p-value or critical region:



p-value for 1-tailed test < 0.05
⇒ p-value for 2-tailed test
= 2 × p-value for 1-tailed test
< 0.1



H_0 is rejected under 1-tailed test
⇒ test stat is inside critical region of 1-tailed test at 5%
⇒ test stat is inside critical region of 2-tailed test at 10%

H_0 would be rejected and the same conclusion would be obtained.

Example (2-tailed test): A machine was calibrated to manufacture 1-metre rods. The lengths x of a sample of 100 rods were recorded. It was found that $\Sigma x = 90$ and $\Sigma x^2 = 92$.

(i) Test, at the 5% level, whether the machine is producing rods of the wrong length.

(ii) Another sample of n rods was tested and it was concluded at the 5% level that the rods were of the wrong length. Using the values of $\bar{x}$ and s^2 as above, find the least value of n.

(iii) $H_0 : \mu = \mu_0$ was tested against $H_1 : \mu \neq \mu_0$ using a sample of 100 rods. Using the values of $\bar{x}$ and s^2 above, find the range of possible values of μ_0 if H_0 is not rejected at the 5% level.

$$\bar{x} = \frac{90}{100} = 0.9$$

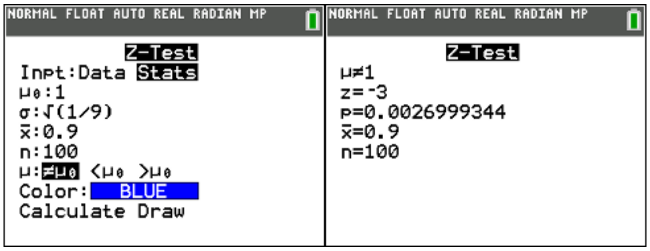
$$s^2 = \frac{1}{99}\left[92 - \frac{90^2}{100}\right] = \frac{1}{9}.$$

(i) $H_0 : \mu = 1$

$H_1 : \mu \neq 1$

Under H_0, since $n = 100$ is large, by CLT, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ approximately.

Test statistic $Z = \dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim N(0, 1)$.

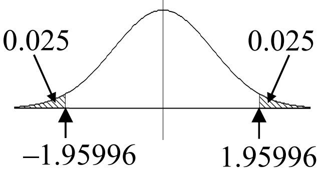


```
NORMAL FLOAT AUTO REAL RADIAN MP
Z-Test
Inpt:Data Stats
μ0:1
σ:√(1/9)
x̄:0.9
n:100
μ:≠μ0 <μ0 >μ0
Color: BLUE
Calculate Draw
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
Z-Test
μ≠1
z=-3
p=0.0026999344
x̄=0.9
n=100
```

Since p-value = 0.00270 < 0.05, we reject H_0. There is sufficient evidence at 5% level of significance to say that the machine is producing rods of the wrong length.

(iii) H_0 is rejected $\Rightarrow \dfrac{0.9-1}{\sqrt{\frac{1/9}{n}}} \leq -1.95996$ or $\dfrac{0.9-1}{\sqrt{\frac{1/9}{n}}} \geq 1.95996$



$-0.1\sqrt{n} \leq -0.65332$ or $-0.1\sqrt{n} \geq 0.65332$ (reject)

$\sqrt{n} \geq 6.5332$

$n \geq 42.68$

$\therefore$ least $n = 43$

(iv) H_0 is not rejected $\Rightarrow -1.95996 < \dfrac{0.9-\mu_0}{\sqrt{\frac{1/9}{100}}} < 1.95996$

$$-0.065332 < 0.9 - \mu_0 < 0.065332$$

$$0.835 < \mu_0 < 0.965$$

SUMMARY OF CORRELATION AND REGRESSION

Formulas given in MF26: Estimated product moment correlation coefficient:

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\sqrt{\Sigma(y-\bar{y})^2}} = \frac{\Sigma xy - \frac{\Sigma x\,\Sigma y}{n}}{\sqrt{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}\sqrt{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}}$$

Estimated regression line of y on x is $y - \bar{y} = b(x - \bar{x})$, where $b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$

Formulas NOT given in MF26:

Estimated regression coefficient $b = \frac{\Sigma xy - \frac{\Sigma x\,\Sigma y}{n}}{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}$

Estimated regression line of x on y is $x - \bar{x} = d(y - \bar{y})$, where $d = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{\Sigma xy - \frac{\Sigma x\,\Sigma y}{n}}{\Sigma y^2 - \frac{(\Sigma y)^2}{n}}$

If x is the independent variable, then we use the regression line of y on x (to estimate x or y).
If y is the independent variable, then we use the regression line of x on y (to estimate x or y).
Otherwise if there are no obvious independent variables, we use the regression line of y on x to estimate y, and the regression line of x on y to estimate x.

Given $n = 3$, $\Sigma x = 6$, $\Sigma y = 10$, $\Sigma x^2 = 14$, $\Sigma y^2 = 36$, $\Sigma xy = 22$, find
(i) the product moment correlation coefficient,
(ii) the linear regression line of y on x,
(iii) the linear regression line of x on y.

(i) $$r = \frac{22 - \frac{6(10)}{3}}{\sqrt{14 - \frac{6^2}{3}}\sqrt{36 - \frac{10^2}{3}}} = \frac{2}{\sqrt{2}\sqrt{8/3}} = \frac{\sqrt{3}}{2}$$

(ii) $$b = \frac{22 - \frac{6(10)}{3}}{14 - \frac{6^2}{3}} = \frac{2}{2} = 1$$

$$y - \frac{10}{3} = 1(x-2)$$

$\therefore$ regression line of y on x is $y = \frac{4}{3} + x$.

(iii) $$d = \frac{22 - \frac{6(10)}{3}}{36 - \frac{10^2}{3}} = \frac{2}{8/3} = \frac{3}{4}$$

$$x - 2 = \frac{3}{4}\left(y - \frac{10}{3}\right)$$

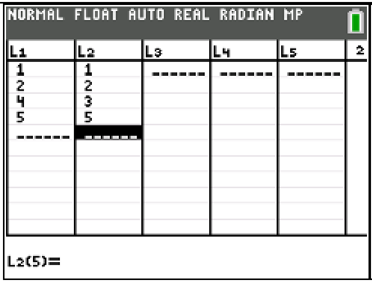
$\therefore$ regression line of x on y is $x = \frac{3}{4}y - \frac{1}{2}$.

Given the following data:

x	1	2	4	5
y	1	2	3	5

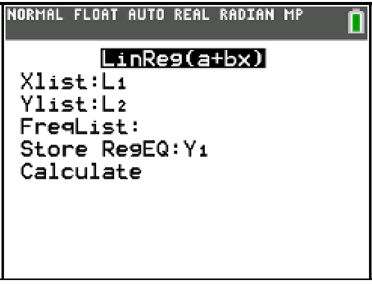
(i) Find the product moment correlation coefficient.
(ii) Find the linear regression line of y on x in the form $y = a + bx$.
Plot a scatter diagram for the data together with the regression line $y = a + bx$.
(iii) Find the linear regression line of x on y in the form $x = c + dy$.
(iv) Use the appropriate regression line to estimate the value of x when $y = 4$.
Comment on the reliability of the estimate.
(v) For the model $y = ab^x$, find the regression line of ln y on x, giving y in terms of x.

(i)

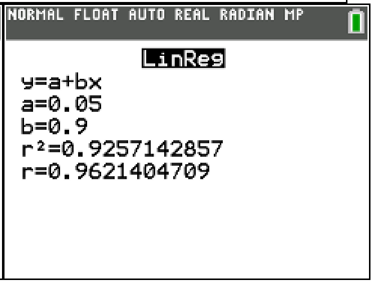


```
NORMAL FLOAT AUTO REAL RADIAN MP
L1     L2     L3     L4     L5     2
1      1      ------ ------ ------
2      2
4      3
5      5
------ ------

L2(5)=
```



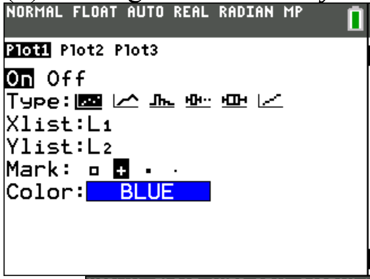
```
NORMAL FLOAT AUTO REAL RADIAN MP
 LinReg(a+bx)
Xlist:L1
Ylist:L2
FreqList:
Store RegEQ:Y1
Calculate
```



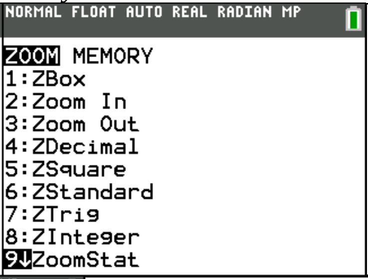
```
NORMAL FLOAT AUTO REAL RADIAN MP
   LinReg
y=a+bx
a=0.05
b=0.9
r²=0.9257142857
r=0.9621404709
```

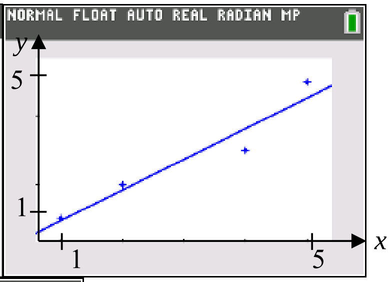
Correlation Coefficient $r = 0.962$

(ii) Regression line of y on x is $y = 0.05 + 0.9x$.

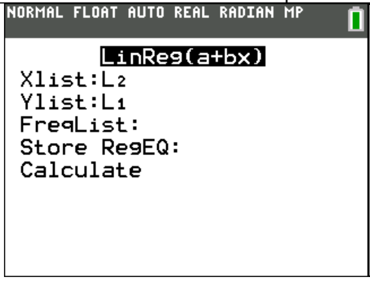


```
NORMAL FLOAT AUTO REAL RADIAN MP
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Ylist:L2
Mark: □ + · .
Color: BLUE
```

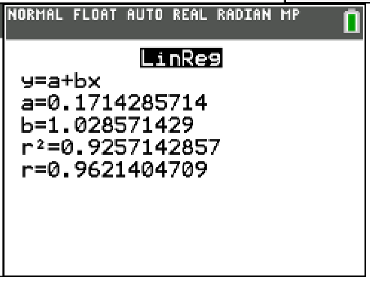




(iii)



```
NORMAL FLOAT AUTO REAL RADIAN MP
     LinReg(a+bx)
Xlist:L2
Ylist:L1
FreqList:
Store RegEQ:
Calculate
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
       LinReg
y=a+bx
a=0.1714285714
b=1.028571429
r²=0.9257142857
r=0.9621404709
```

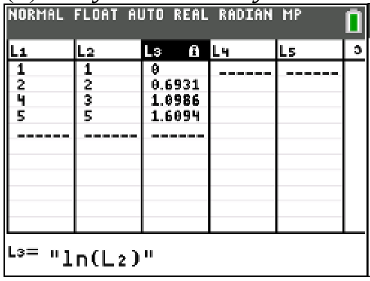
Regression line of x on y is $x = 0.171 + 1.03y$.

(iv) Since there is no clear independent variable, we use the regression line of x on y.

When $y = 4$, $x \approx 0.17143 + 1.02857(4) = 4.28$

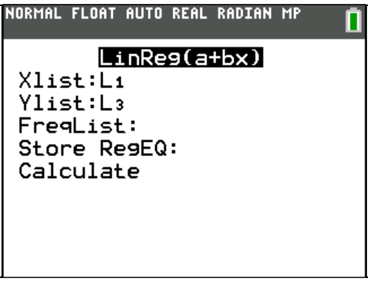
Since $| r | = 0.962$ is close to 1 and $y = 4$ is within the data range, the estimate is reliable.

(v) $y = ab^x \Rightarrow \ln y = \ln a + x \ln b$

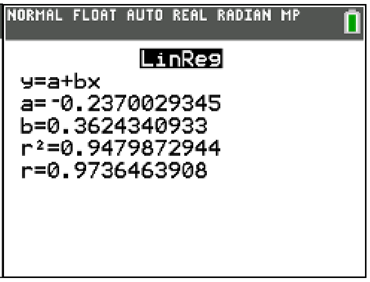


```
NORMAL FLOAT AUTO REAL RADIAN MP
L1     L2     L3     L4     L5     3
1      1      0      ------ ------
2      2      0.6931
4      3      1.0986
5      5      1.6094
------ ------ ------

L3= "ln(L2)"
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
     LinReg(a+bx)
Xlist:L1
Ylist:L3
FreqList:
Store RegEQ:
Calculate
```



```
NORMAL FLOAT AUTO REAL RADIAN MP
       LinReg
y=a+bx
a=-0.2370029345
b=0.3624340933
r²=0.9479872944
r=0.9736463908
```

Regression line of ln y on x is $\ln y = -0.23700 + 0.36243x$

i.e. $y = e^{-0.23700} e^{0.36243x} = 0.789(1.44^x)$