NON-PARAMETRIC TESTS

These tests are useful when the sample data consist only of ranks of the observations, or when the data do not satisfy the assumptions of the corresponding parametric tests.

Comparing 2 Populations using Independent Samples: the MANN-WHITNEY U TEST (also known as the WILCOXON RANK SUM TEST)

We first rank all the sample observations from the smallest to the largest, and compute the rank sum for each sample. Ties are treated by averaging the tied ranks.

H_0	H_1	Test Statistic	Reject H_0 if
Probability distributions 1 & 2 are identical	Probability distribution 1 is either to the left or right of probability distribution 2	$u = n_1n_2 + \frac{n_1}{2}(n_1 + 1) - T_1$ or $u = n_1n_2 + \frac{n_2}{2}(n_2 + 1) - T_2$ whichever is smaller	$P(U \leq u) < \frac{\alpha}{2}$
	Probability distribution 1 is to the right of probability distribution 2	$u = n_1n_2 + \frac{n_1}{2}(n_1 + 1) - T_1$	$P(U \leq u) < \alpha$
	Probability distribution 1 is to the left of probability distribution 2	$u = n_1n_2 + \frac{n_2}{2}(n_2 + 1) - T_2$	$P(U \leq u) < \alpha$

Example:
We claim that boys can do more chin-ups than girls. The chin-up scores of 2 samples of children are as follows.

Chin-up Scores of Girls	Rank	Chin-up Scores of Boys	Rank
1	1	4	5
2	2	4	5
3	3	5	7.5
4	5	5	7.5
	T_1 = 11		**T_2 = 25**

Do the samples support our claim? Use a significance level of $\alpha = 0.05$.

$$u = (4)(4) + \frac{4}{2}(5) - 25 = 16 + 10 - 25 = 1$$

Since $P(U \leq 1) = 0.0286 < 0.05$, we have **sufficient evidence** at 5% level of significance to conclude that boys can do **significantly more chin-ups** than girls.

Comparing 2 Populations using Matched Pairs: the WILCOXON SIGNED RANK TEST

We first find the differences between the matched pairs of observations. Differences equal to 0 are eliminated and the number of differences n is reduced accordingly. Then we rank all the absolute values of the differences. Ties are treated by averaging the tied ranks.

H_0	H_1	Test Statistic	Reject H_0 if
Probability distributions 1 & 2 are identical	Probability distribution 1 is either to the left or right of probability distribution 2	$T = T^-$ or T^+ whichever is smaller	$T \leq T_{\alpha,n}$
	Probability distribution 1 is to the right of probability distribution 2	T^- = rank sum of the negative differences	$T^- \leq T_{\alpha,n}$
	Probability distribution 1 is to the left of probability distribution 2	T^+ = rank sum of the positive differences	$T^+ \leq T_{\alpha,n}$

Example:
We claim that children can do more chin-ups after taking a special diet. A random sample of

children produced the following data:

	No. of chin-ups		Difference	Absolute difference	Rank	
	Before	After				
Ann	5	4	1	1	1	←
Bob	4	6	–2	2	2.5	
Clare	3	1	2	2	2.5	←
Dave	1	5	–4	4	4.5	
Eve	2	6	–4	4	4.5	

Do the data support our claim? Use a significance level of $\alpha = 0.05$.

$T^+ = 1 + 2.5 = 3.5$

Since 3.5 > the critical value $T_{0.05,5} = 1$, we have **insufficient evidence** at 5% level of significance to conclude that children can do **significantly more chin-ups** after taking the diet.

Comparing k Populations using Independent Samples: KRUSKAL-WALLIS H TEST

We first rank all the n sample observations from the smallest to the largest, and compute the rank sum T_i for each sample. Ties are treated by averaging the tied ranks.

H_0 : the k probability distributions are identical

H_1 : at least 2 of the probability distributions differ in location

Test statistic $H = \frac{12}{n(n+1)}\left(\sum_{i=1}^{k}\frac{T_i^2}{n_i}\right) - 3(n+1)$ where n_i = size of sample i

We reject H_0 if $H > \chi^2_{\alpha,k-1}$

Example:

The chin-up scores of 12 children taking 3 types of diets are as follows:

Vegetarian Diet		High Protein Diet		Normal Diet	
Score	**Rank**	**Score**	**Rank**	**Score**	**Rank**
1	1	2	3	3	6.5
2	3	4	10	3	6.5
2	3	4	10	3	6.5
3	6.5	5	12	4	10
	T_1 = 13.5		**T_2 = 35**		**T_3 = 29.5**

Does the diet make a difference in the score obtained?

$$H = \frac{12}{(12)(13)}\left(\frac{13.5^2}{4} + \frac{35^2}{4} + \frac{29.5^2}{4}\right) - 3(13) = \frac{1}{13}\left(45\frac{9}{16} + 306\frac{1}{4} + 217\frac{9}{16}\right) - 39 = 4.7981$$

Since 4.7981 < the critical value $\chi^2_{0.05,2} = 5.99147$, we have **insufficient evidence** at 5% level of significance to conclude that **the diet make a difference in the score** obtained.

Comparing k Populations using Randomised Block Design: the FRIEDMAN F_r TEST

We first rank the k observations in each block from the smallest to the largest, and compute the rank sum T_i for each of the k treatments. Ties are treated by averaging the tied ranks.

H_0 : the k probability distributions are identical

H_1 : at least 2 of the probability distributions differ in location

Test statistic $F_r = \frac{12}{bk(k+1)}\left(\sum_{i=1}^{k} T_i^2\right) - 3b(k+1)$ where b = number of blocks

We reject H_0 if $F_r > \chi^2_{\alpha,k-1}$

Example:

The chin-up scores of 4 children when taking different types of diets are as follows:

	Vegetarian Diet		High Protein Diet		Normal Diet	
	Score	**Rank**	**Score**	**Rank**	**Score**	**Rank**
Alan	1	1	2	2	3	3
Bob	2	1	4	3	3	2
Carl	2	1	4	3	3	2
Dave	3	1	5	3	4	2
Sum		**$T_1 = 4$**		**$T_2 = 11$**		**$T_3 = 9$**

Does the diet make a difference in the score obtained?

Test statistic $F_r = \frac{12}{(4)(3)(4)}\left(4^2 + 11^2 + 9^2\right) - 3(4)(4) = 54.5 - 48 = 6.5$

Since 6.5 > the critical value $\chi^2_{0.05,2} = 5.99147$, we have **sufficient evidence** at 5% level of significance to conclude that **the diet does make a difference** in the score obtained.

Testing correlation between 2 variables using ranked data: the SPEARMAN'S RANK CORRELATION COEFFICIENT r_s

We first rank the values of each of the variables separately. Ties are treated by averaging the tied ranks. We then compute $r_s = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$ where

$SS_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$, $SS_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$, $SS_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$

This is exactly the same formula as for the simple correlation coefficient r, except that the values of x and y here denote the rank of the raw data rather than the raw data themselves.

When there are no ties, the formula is simplified to $r_s = 1 - \frac{6\,\Sigma d^2}{n\,(n^2 - 1)}$ where $d = x - y$.

H_0	H_1	Reject H_0 if
There is no correlation between the ranked pairs	Ranked pairs are correlated	$r_s \le -r_{\alpha/2,n}$ or $r_s \ge r_{\alpha/2,n}$
	Ranked pairs are positively correlated	$r_s \ge r_{\alpha,n}$
	Ranked pairs are negatively correlated	$r_s \le -r_{\alpha,n}$

Example:

The ages and chin-up scores of 4 children are as follows:

age	rank x	chin-up scores	rank y
8	1	1	1.5
9	2	1	1.5
11	3	3	3
12	4	5	4
	$\Sigma x = 10$		**$\Sigma y = 10$**

xy	x^2	y^2
1.5	1	2.25
3	4	2.25
9	9	9
16	16	16
$\Sigma xy = 29.5$	**$\Sigma x^2 = 30$**	**$\Sigma y^2 = 29.5$**

Are x and y positively correlated? Use a significance level of $\alpha = 0.05$.

$$SS_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n} = 29.5 - \frac{(10)(10)}{4} = 29.5 - 25 = 4.5$$

$$SS_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n} = 30 - \frac{10^2}{4} = 30 - 25 = 5$$

$$SS_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n} = 29.5 - \frac{10^2}{4} = 29.5 - 25 = 4.5$$

Correlation coefficient $r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}} = \frac{4.5}{\sqrt{(5)(4.5)}} = 0.949$

Since 0.949 > the critical value $r_{0.05,5} = 0.9$, we have **sufficient evidence** at 5% level of significance to conclude that x and y are **positively correlated**.

Calculating the Critical Values for Wilcoxon Signed Rank Test

T_+ = sum of positive ranks, T_- = sum of negative ranks

Probability Distribution of T_+ and T_- under H_0 for n = 3, assuming no ties.

			t_+	$P(T_+ = t_+)$	t_-	$P(T_- = t_-)$
–1	–2	–3	0	$\frac{1}{8}$	6	$\frac{1}{8}$
1	–2	–3	1	$\frac{1}{8}$	5	$\frac{1}{8}$
–1	2	–3	2	$\frac{1}{8}$	4	$\frac{1}{8}$
–1	–2	3	3	$\frac{1}{4}$	3	$\frac{1}{4}$
1	2	–3				
1	–2	3	4	$\frac{1}{8}$	2	$\frac{1}{8}$
–1	2	3	5	$\frac{1}{8}$	1	$\frac{1}{8}$
1	2	3	6	$\frac{1}{8}$	0	$\frac{1}{8}$

For n = 3, p-value $\geq \frac{1}{8}$ = 0.125 for 1-tailed test (or $\geq \frac{1}{4}$ = 0.25 for 2-tailed test).

So we do not reject H_0 for all levels of significance < 12.5% for 1-tailed test (or < 25% for 2-tailed test).

Probability Distribution of T_+ and T_- under H_0 for n = 4, assuming no ties.

				t_+	$P(T_+ = t_+)$	t_-	$P(T_- = t_-)$
–1	–2	–3	–4	0	$\frac{1}{16}$	10	$\frac{1}{16}$
1	–2	–3	–4	1	$\frac{1}{16}$	9	$\frac{1}{16}$
–1	2	–3	–4	2	$\frac{1}{16}$	8	$\frac{1}{16}$
–1	–2	3	–4	3	$\frac{1}{8}$	7	$\frac{1}{8}$
1	2	–3	–4				
–1	–2	–3	4	4	$\frac{1}{8}$	6	$\frac{1}{8}$
1	–2	3	–4				
1	–2	–3	4	5	$\frac{1}{8}$	5	$\frac{1}{8}$
–1	2	3	–4				
–1	2	–3	4	6	$\frac{1}{8}$	4	$\frac{1}{8}$
1	2	3	–4				
–1	–2	3	4	7	$\frac{1}{8}$	3	$\frac{1}{8}$
1	2	–3	4				
1	–2	3	4	8	$\frac{1}{16}$	2	$\frac{1}{16}$
–1	2	3	4	9	$\frac{1}{16}$	1	$\frac{1}{16}$
1	2	3	4	10	$\frac{1}{16}$	0	$\frac{1}{16}$

For n = 4, p-value $\geq \frac{1}{16}$ = 0.0625 for 1-tailed test (or $\geq \frac{1}{8}$ = 0.125 for 2-tailed test).

So we do not reject H_0 for all levels of significance < 6.25% for 1-tailed test (or < 12.5% for 2-tailed test).

Probability Distribution of T_+ and T_- under H_0 for n = 5, assuming no ties.

					t_+	$P(T_+ = t_+)$	t_-	$P(T_- = t_-)$
–1	–2	–3	–4	–5	0	$\frac{1}{32}$	15	$\frac{1}{32}$
1	–2	–3	–4	–5	1	$\frac{1}{32}$	14	$\frac{1}{32}$
–1	2	–3	–4	–5	2	$\frac{1}{32}$	13	$\frac{1}{32}$
–1	–2	3	–4	–5	3	$\frac{1}{16}$	12	$\frac{1}{16}$
1	2	–3	–4	–5				
–1	–2	–3	4	–5	4	$\frac{1}{16}$	11	$\frac{1}{16}$
1	–2	3	–4	–5				
–1	–2	–3	–4	5	5	$\frac{3}{32}$	10	$\frac{3}{32}$
1	–2	–3	4	–5				
–1	2	3	–4	–5				
1	–2	–3	–4	5	6	$\frac{3}{32}$	9	$\frac{3}{32}$
–1	2	–3	4	–5				
1	2	3	–4	–5				
–1	2	–3	–4	5	7	$\frac{3}{32}$	8	$\frac{3}{32}$
–1	–2	3	4	–5				
1	2	–3	4	–5				
–1	–2	3	–4	5	8	$\frac{3}{32}$	7	$\frac{3}{32}$
1	2	–3	–4	5				
1	–2	3	4	–5				
–1	–2	–3	4	5	9	$\frac{3}{32}$	6	$\frac{3}{32}$
1	–2	3	–4	5				
–1	2	3	4	–5				
1	–2	–3	4	5	10	$\frac{3}{32}$	5	$\frac{3}{32}$
–1	2	3	–4	5				
1	2	3	4	–5				
–1	2	–3	4	5	11	$\frac{1}{16}$	4	$\frac{1}{16}$
1	2	3	–4	5				
–1	–2	3	4	5	12	$\frac{1}{16}$	3	$\frac{1}{16}$
1	2	–3	4	5				
1	–2	3	4	5	13	$\frac{1}{32}$	2	$\frac{1}{32}$
–1	2	3	4	5	14	$\frac{1}{32}$	1	$\frac{1}{32}$
1	2	3	4	5	15	$\frac{1}{32}$	0	$\frac{1}{32}$

For n = 5, $P(T_+ = 0) = \frac{1}{32} = 0.03125 < 0.05$.

$P(T_+ \le 1) = P(T_+ = 0) + P(T_+ = 1) = \frac{1}{16} = 0.0625 > 0.05$.

We reject H_0 if test statistic = 0 at 5% level. So critical value for 5% level = 0.

Probability Distribution of T_+ and T_- under H_0 for n = 6, assuming no ties.

						t_+	$P(T_+ = t_+)$	t_-	$P(T_- = t_-)$
–1	–2	–3	–4	–5	–6	0	$\frac{1}{64}$	21	$\frac{1}{64}$
1	–2	–3	–4	–5	–6	1	$\frac{1}{64}$	20	$\frac{1}{64}$
–1	2	–3	–4	–5	–6	2	$\frac{1}{64}$	19	$\frac{1}{64}$
–1	–2	3	–4	–5	–6	3	$\frac{1}{32}$	18	$\frac{1}{32}$
1	2	–3	–4	–5	–6				
–1	–2	–3	4	–5	–6	4	$\frac{1}{32}$	17	$\frac{1}{32}$
1	–2	3	–4	–5	–6				
–1	–2	–3	–4	5	–6	5	$\frac{3}{64}$	16	$\frac{3}{64}$
1	–2	–3	4	–5	–6				
–1	2	3	–4	–5	–6				
–1	–2	–3	–4	–5	6	6	$\frac{1}{16}$	15	$\frac{1}{16}$
1	–2	–3	–4	5	–6				
–1	2	–3	4	–5	–6				
1	2	3	–4	–5	–6				
1	–2	–3	–4	–5	6	7	$\frac{1}{16}$	14	$\frac{1}{16}$
–1	2	–3	–4	5	–6				
–1	–2	3	4	–5	–6				
1	2	–3	4	–5	–6				
–1	2	–3	–4	–5	6	8	$\frac{1}{16}$	13	$\frac{1}{16}$
–1	–2	3	–4	5	–6				
1	2	–3	–4	5	–6				
1	–2	3	4	–5	–6				
–1	–2	3	–4	–5	6	9	$\frac{5}{64}$	12	$\frac{5}{64}$
–1	–2	–3	4	5	–6				
1	2	–3	–4	–5	6				
1	–2	3	–4	5	–6				
–1	2	3	4	–5	–6				
–1	–2	–3	4	–5	6	10	$\frac{5}{64}$	11	$\frac{5}{64}$
1	–2	3	–4	–5	6				
1	–2	–3	4	5	–6				
–1	2	3	–4	5	–6				
1	2	3	4	–5	–6				
						11	:	:	:

For n = 6, $P(T_+ = 0) = \frac{1}{64} = 0.015625 < 0.025$.

$P(T_+ \le 1) = P(T_+ = 0) + P(T_+ = 1) = \frac{1}{32} = 0.03125 > 0.025$.

We reject H_0 if test statistic ≤ 0 at 2.5% level. So critical value for 2.5% level = 0.

$P(T_+ \le 2) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) = \frac{3}{64} = 0.046875 < 0.05$.

$P(T_+ \le 3) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) + P(T_+ = 3) = \frac{5}{64} = 0.078125 > 0.05$.

We reject H_0 if test statistic ≤ 2 at 5% level. So critical value for 5% level = 2.

Probability Distribution of T_+ and T_- under H_0 for n = 7, assuming no ties.

							t_+	$P(T_+ = t_+)$	t_-	$P(T_- = t_-)$
–1	–2	–3	–4	–5	–6	–7	0	$\frac{1}{128}$	28	$\frac{1}{128}$
1	–2	–3	–4	–5	–6	–7	1	$\frac{1}{128}$	27	$\frac{1}{128}$
–1	2	–3	–4	–5	–6	–7	2	$\frac{1}{128}$	26	$\frac{1}{128}$
–1	–2	3	–4	–5	–6	–7	3	$\frac{1}{64}$	25	$\frac{1}{64}$
1	2	–3	–4	–5	–6	–7				
–1	–2	–3	4	–5	–6	–7	4	$\frac{1}{64}$	24	$\frac{1}{64}$
1	–2	3	–4	–5	–6	–7				
							:	:	:	:

For n = 7, $P(T_+ = 0) = \frac{1}{128} = 0.0078125 < 0.01$.

$P(T_+ \le 1) = P(T_+ = 0) + P(T_+ = 1) = \frac{1}{64} = 0.015625 > 0.01$.

We reject H_0 if test statistic ≤ 0 at 1% level. So critical value for 1% level = 0.

$P(T_+ \le 2) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) = \frac{3}{128} = 0.0234375 < 0.025$.

$P(T_+ \le 3) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) + P(T_+ = 3) = \frac{5}{128} = 0.0390625 > 0.025$.

We reject H_0 if test statistic ≤ 2 at 2.5% level. So critical value for 2.5% level = 2.

$P(T_+ \le 4) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 4) = \frac{7}{128} = 0.0546875 > 0.05$.

We reject H_0 if test statistic ≤ 3 at 5% level. So critical value for 5% level = 3.

Probability Distribution of T_+ and T_- under H_0 for n = 8, assuming no ties.

								t_+	$P(T_+ = t_+)$	t_-	$P(T_- = t_-)$
–1	–2	–3	–4	–5	–6	–7	–8	0	$\frac{1}{256}$	36	$\frac{1}{256}$
1	–2	–3	–4	–5	–6	–7	–8	1	$\frac{1}{256}$	35	$\frac{1}{256}$
–1	2	–3	–4	–5	–6	–7	–8	2	$\frac{1}{256}$	34	$\frac{1}{256}$
–1	–2	3	–4	–5	–6	–7	–8	3	$\frac{1}{128}$	33	$\frac{1}{128}$
1	2	–3	–4	–5	–6	–7	–8				
–1	–2	–3	4	–5	–6	–7	–8	4	$\frac{1}{128}$	32	$\frac{1}{128}$
1	–2	3	–4	–5	–6	–7	–8				
–1	–2	–3	–4	5	–6	–7	–8	5	$\frac{3}{256}$	31	$\frac{3}{256}$
1	–2	–3	4	–5	–6	–7	–8				
–1	2	3	–4	–5	–6	–7	–8				
–1	–2	–3	–4	–5	6	–7	–8	6	$\frac{1}{64}$	30	$\frac{1}{64}$
1	–2	–3	–4	5	–6	–7	–8				
–1	2	–3	4	–5	–6	–7	–8				
1	2	3	-4	–5	–6	–7	–8				
								:			

For n = 8, $P(T_+ = 0) = \frac{1}{256} = 0.00390625 < 0.005$.

$P(T_+ \le 1) = P(T_+ = 0) + P(T_+ = 1) = \frac{1}{128} = 0.0078125 > 0.005$.

We reject H_0 if test statistic = 0 at 0.5% level. So critical value for 0.5% level = 0.

$P(T_+ \leq 2) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) = \frac{3}{256} = 0.01171875 > 0.01$.

We reject H_0 if test statistic ≤ 1 at 1% level. So critical value for 1% level = 1.

$P(T_+ \leq 3) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) + P(T_+ = 3) = \frac{5}{256} = 0.01953125 > 0.025$.

$P(T_+ \leq 4) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 4) = \frac{7}{256} = 0.02734375 > 0.025$.

We reject H_0 if test statistic ≤ 3 at 2.5% level. So critical value for 2.5% level = 3.

$P(T_+ \leq 5) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 5) = \frac{5}{128} = 0.0390625 < 0.05$.

$P(T_+ \leq 6) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 6) = \frac{7}{128} = 0.0546875 > 0.05$.

We reject H_0 if test statistic ≤ 5 at 5% level. So critical value for 5% level = 5.

Probability Distribution of T_+ and T_- under H_0 for n = 9, assuming no ties.

									t_+	$P(T_+ = t_+)$	t_-	$P(T_- = t_-)$
–1	–2	–3	–4	–5	–6	–7	–8	–9	0	$\frac{1}{512}$	45	$\frac{1}{512}$
1	–2	–3	–4	–5	–6	–7	–8	–9	1	$\frac{1}{512}$	44	$\frac{1}{512}$
–1	2	–3	–4	–5	–6	–7	–8	–9	2	$\frac{1}{512}$	43	$\frac{1}{512}$
–1	–2	3	–4	–5	–6	–7	–8	–9	3	$\frac{1}{256}$	42	$\frac{1}{256}$
1	2	–3	–4	–5	–6	–7	–8	–9				
–1	–2	–3	4	–5	–6	–7	–8	–9	4	$\frac{1}{256}$	41	$\frac{1}{256}$
1	–2	3	–4	–5	–6	–7	–8	–9				
–1	–2	–3	–4	5	–6	–7	–8	–9	5	$\frac{3}{512}$	40	$\frac{3}{512}$
1	–2	–3	4	–5	–6	–7	–8	–9				
–1	2	3	–4	–5	–6	–7	–8	–9				
–1	–2	–3	–4	–5	6	–7	–8	–9	6	$\frac{1}{128}$	39	$\frac{1}{128}$
1	–2	–3	–4	5	–6	–7	–8	–9				
–1	2	–3	4	–5	–6	–7	–8	–9				
1	2	3	-4	–5	–6	–7	–8	–9				
–1	–2	–3	–4	–5	–6	7	–8	–9	7	$\frac{5}{512}$	38	$\frac{5}{512}$
1	–2	–3	–4	–5	6	–7	–8	–9				
–1	2	–3	–4	5	–6	–7	–8	–9				
–1	–2	3	4	–5	–6	–7	–8	–9				
1	2	-3	4	–5	–6	–7	–8	–9				
–1	–2	–3	–4	–5	–6	–7	8	–9	8	$\frac{3}{256}$	37	$\frac{3}{256}$
1	–2	–3	–4	–5	–6	7	–8	–9				
–1	2	–3	–4	–5	6	–7	–8	–9				
–1	–2	3	–4	5	–6	–7	–8	–9				
1	2	3	4	5	–6	–7	–8	–9				
1	–2	3	4	-5	–6	–7	–8	–9				
–1	–2	–3	–4	–5	–6	–7	–8	9	9	$\frac{1}{64}$	36	$\frac{1}{64}$
1	–2	–3	–4	–5	–6	–7	8	–9				
–1	2	–3	–4	–5	–6	7	–8	–9				
–1	–2	3	–4	–5	6	–7	–8	–9				
1	2	–3	-4	-5	6	–7	–8	–9				
–1	–2	–3	4	5	–6	–7	–8	–9				
1	–2	3	-4	5	–6	–7	–8	–9				
-1	2	3	4	-5	–6	–7	–8	–9				

	:			

For n = 9, $P(T_+ \leq 1) = P(T_+ = 0) + P(T_+ = 1) = \frac{1}{256} = 0.00390625 < 0.005$.

$P(T_+ \leq 2) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) = \frac{3}{512} = 0.005859375 > 0.005$.

We reject H_0 if test statistic ≤ 1 at 0.5% level. So critical value for 0.5% level = 1.

$P(T_+ \leq 3) = P(T_+ = 0) + P(T_+ = 1) + P(T_+ = 2) + P(T_+ = 3) = \frac{5}{512} = 0.009765625 < 0.01$.

$P(T_+ \leq 4) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 4) = \frac{7}{512} = 0.013671875 > 0.01$.

We reject H_0 if test statistic ≤ 3 at 1% level. So critical value for 1% level = 3.

$P(T_+ \leq 5) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 5) = \frac{5}{256} = 0.01953125 < 0.025$.

$P(T_+ \leq 6) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 6) = \frac{7}{256} = 0.02734375 > 0.025$.

We reject H_0 if test statistic ≤ 5 at 2.5% level. So critical value for 2.5% level = 5.

$P(T_+ \leq 8) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 8) = \frac{25}{512} = 0.048828125 < 0.05$.

$P(T_+ \leq 9) = P(T_+ = 0) + P(T_+ = 1) + ... + P(T_+ = 9) = \frac{33}{512} = 0.064453125 > 0.05$.

We reject H_0 if test statistic ≤ 8 at 5% level. So critical value for 5% level = 8.